Partial Riemann problem, boundary conditions, and gas dynamics

François Dubois

Conservatoire National des Arts et Métiers
Department of Mathematics, Paris, France.

Applications Scientifiques du Calcul Intensif
bât. 506, BP 167, F-91 403 Orsay Cedex, France.
dubois@asci.fr

June 29, 2000 □

Keywords: hyperbolic systems, finite volumes.

AMS classification: 35L04, 35L60, 35Q35, 65N08.

□ Published in *Absorbing Boundaries and Layers, Domain Decomposition Methods: Applications to Large Scale Computations*, pages 16-77, Edited by Loïc Tourrette and Laurence Halpern, Nova Science Publishers, Inc, New York, 2001, ISBN: 1-56072-940-6.
Révision 10 May 2002. Edition 11 January 2011.

Contents

0) Introduction.

• We study in this article the problem of boundary conditions for the equations of gas dynamics. We restrict ourselves to the model of perfect fluid and essentially to simple unidimensional geometry. The problem can be considered from two points of view. First the linearization of the Euler equations of

gas dynamics allows the development of the so-called method of characteristics and the linearized problem is mathematically well posed when boundary data are associated with the characteristics that comes inside the domain of study. Second the physical approach separates clearly fluid boundary conditions where incomplete physical data are given, and rigid walls where the slip condition describes the interaction between the fluid and a given object located inside or around the fluid. There is at our knowledge no simple way to connect these mathematical and physical approaches for strongly nonlinear interactions.

• We introduce in this contribution the notion of partial Riemann problem. Recall that the Riemann problem describes a shock tube interaction between two given states ; the partial Riemann problem is a generalization of the previous concept and introduces the notion of boundary manifold. In what follows, we first recall very classical notions concerning gas dynamics and the associated Riemann problem. In a second part, we introduce the partial Riemann problem for general systems of conservation laws and proves that this problem admits a solution in some class of appropriate nonlinear waves. In section 3, we recall the linearized analysis with the method of characteristics, introduce the weak formulation of the Dirichlet boundary condition for nonlinear situations in terms of the partial Riemann problem and show that lot of physically relevant situations are described with this theoretical framework. In the last paragraph, we propose a practical implementation of the previous considerations with the finite volume method.

1) Euler equations of gas dynamics.

1.1 Thermodynamics.

• We study a perfect gas submitted to a motion with variable velocity in space and time. We note first that the primitive unknowns of this problem are the scalar fields that characterize the thermodynamics of the gas, *i.e.* density ρ, internal energy e, temperature T, and pressure p (see *e.g.* the book of Callen [Ca85]). In what follows, we suppose that the gas is a polytropic perfect gas ; it has constant specific heats at constant volume C_v and at constant pressure C_p. These two quantities do not depend on any thermodynamic variable like temperature or pressure ; we denote by γ their ratio :

(1.1.1) $$\gamma \,=\, \frac{C_p}{C_v}\,(=\, Cste)\,.$$

We suppose that the gas satisfies the law of perfect gas that can be written with the following form :

(1.1.2) $$p \,=\, (\gamma-1)\,\rho\,e\,.$$

As usual, internal energy and temperature are linked together by the Joule-Thomson relation :

(1.1.3) $\quad e \;=\; C_v \, T \,.$

• In the formalism proposed by Euler during the 18^0 century, the motion is described with the help of an unknown vector field u which is a function of space x and time t :

(1.1.4) $\quad u \;=\; u(x,\, t) \,.$

In the following, we will suppose that space x has only one dimemsion ($x \in \mathbb{R}$). We have four unknown functions (density, velocity, pressure and internal energy) linked together by the state law (1.1.2) In consequence, we need three complementary equations in order to define a unique solution of the problem.

• The general laws of Physics assume that mass, momentum and total energy are conserved quantities, at least in the context of classical physics associated to the paradigm of invariance for the Galileo group of space-time transformations (see *e.g.* Landau and Lifchitz [LL54]). When we write the conservation of mass, momentum and energy inside an infinitesimal volume $\mathrm{d}x$ advected with celerity $u(x,\, t)$, which is exactly the mean velocity of particules that compose the gas, it is classical [LL54] to write the fundamental conservation laws of Physics with the help of divergence operators :

$$\frac{\partial \rho}{\partial t} \;+\; \frac{\partial}{\partial x} \, (\rho \, u) \;=\; 0 \tag{1.1.5}$$

$$\frac{\partial}{\partial t} (\rho \, u) \;+\; \frac{\partial}{\partial x} \, (\rho \, u^2 \,+\, p) \;=\; 0 \tag{1.1.6}$$

$$\frac{\partial}{\partial t} \Big(\frac{1}{2} \rho \, u^2 \,+\, \rho \, e \Big) \;+\; \frac{\partial}{\partial x} \Big(\big(\frac{1}{2} \rho \, u^2 \,+\, p \big) \, u \,+\, p \, u \Big) \;=\; 0 \,. \tag{1.1.7}$$

We introduce the specific total energy E by unity of volume

$$E \;=\; \frac{1}{2} u^2 \,+\, e \tag{1.1.8}$$

and the vector W composed by the "conservative variables" or more precisely by the "conserved variables" :

$$W \;=\; \big(\rho \,,\, \rho \, u \,,\, \rho \, E \big)^{\mathrm{t}} \,. \tag{1.1.9}$$

The conservation laws (1.1.5)-(1.1.7) take the following general form of a so-called system of conservation laws :

$$\frac{\partial W}{\partial t} \;+\; \frac{\partial}{\partial x} \, F(W) \;=\; 0 \tag{1.1.10}$$

where the flux vector $W \longmapsto F(W)$ satisfies the following algebraic expression :

$$F(W) \;=\; \big(\rho \, u \,,\, \rho \, u^2 \,+\, p \,,\, \rho \, u \, E \,+\, p \, u \big)^{\mathrm{t}} \,. \tag{1.1.11}$$

• The system of conservation laws (1.1.10) is an hyperbolic system of equations sometimes stated as a "quasilinear system" of conservation laws. From

a mathematical point of view, the actual state of knowledge (see *e.g.* Serre [Se96]) does not give any answer to the question of the existence of a solution $(x,\,t) \longmapsto W(x,\,t)$ when time t is no longer small enough, even if the initial condition $x \longmapsto W(x,\,0)$ is a regular function of the space variable x. Even if the problem is solved in the scalar case (Kruzkov [Kv70]), nonlinear waves are present in system (1.1.10), they can create discontinuities, and this phenomenon makes the general mathematical study of system (1.1.10) very tricky. Moreover, the unicity of irregular solutions of this kind of system is in general not satisfied. The usual tentative to achieve uniqueness is associated with the incorporation of the second principle of Thermodynamics. Carnot's principle of thermodynamics can be explicitly introduced with the specific entropy s. Recall that for a perfect polytropic gas, the specific entropy is a function of the state thermodynamic variables p and ρ :

$$s \;=\; C_v \,\mathrm{Log}\Big(\frac{\rho_0^\gamma}{p_0}\, \frac{p}{\rho^\gamma} \Big) \tag{1.1.12}$$

and it is easy to see that the following so-called **mathematical entropy** $\eta(\bullet)$

$$\eta(W) \;=\; -\rho\, s \tag{1.1.13}$$

is a convex function of the conservative variables if temperature is positive (we refer *e.g.* to our proof in [Du90]). It admits an entropy flux $\xi(W)$ in the sense of the mathematical theory proposed by Friedrichs and Lax in 1971 [FL71] and we have

$$\xi(W) \;=\; -\rho\, u\, s\,. \tag{1.1.14}$$

The second principle of increasing of physical entropy when time is increasing has been formalized mathematically by Germain-Bader [GB53], Oleinik [Ol57], Godunov [Go61] and Friedrichs-Lax in 1971 [FL71] among others. It takes inside the formalism of mechanics of continuous media the following weak "conservative form" :

$$\frac{\partial}{\partial t}\, \eta(W) \;+\; \frac{\partial}{\partial x}\, \xi(W) \;\leq\; 0\,. \tag{1.1.15}$$

Inequality (1.1.15) is exactly an equality for a regular solution $W(\bullet,\,\bullet)$ of conservation law (1.1.10). It has to be considered in the sense of distributions for a weak solution $W(\bullet,\,\bullet)$ of conservation law (1.1.10).

• The mathematical entropy (1.1.13) is a strictly convex function of the conservative variables if temperature is positive. We introduce the Fréchet derivative $\mathrm{d}\eta(W)$ of entropy $\eta(\bullet)$ with respect to the conservative variables W :

$$\mathrm{d}\eta(W) \;=\; \phi \, \mathrm{d}W \tag{1.1.16}$$

and define the so-called **entropy variables** ϕ. The precise calculus of entropy variables is elementary from the traditional expression of the second principle

of thermodynamics. If a container has volume V, mass M, internal energy $\mathcal{E}$, then entropy S is a **function** $\Sigma(\bullet)$ of the above variables :

(1.1.17) $\qquad S \,=\, \Sigma\,(M,\, V,\, \mathcal{E})\,,$

and moreover, function $\Sigma(\bullet)$ is an extensive function (homogeneous of degree 1) of the preceeding variables, *id est* $\Sigma\,(\lambda\, M,\, \lambda\, V,\, \lambda\, \mathcal{E}) \,=\, \lambda\, \Sigma\,(M,\, V,\, \mathcal{E})\,,$ $\forall\, \lambda \,>\, 0\,.$ The classical differential relation between these quantities can be stated as

(1.1.18) $\qquad {\rm d}\mathcal{E} \,=\, T\,{\rm d}S \,-\, p\,{\rm d}V \,+\, \mu\,{\rm d}M\,.$

Notice that the third intensive thermodynamic variable μ is just the **specific chemical potential**, *i.e.* the chemical potential by unit of mass. Then the global values are related for corresponding specific quantities with the help of the following relations :

(1.1.19) $\qquad M \,=\, \rho\, V\,, \quad \mathcal{E} \,=\, e\, M\,, \quad S \,=\, s\, M\,, \quad \mathcal{E} + p\,V - T\,S \,=\, \mu\, M\,.$

Due to the homogeneity of function $\Sigma(\bullet)$, the mathematical entropy $\eta(\bullet)$ can be simply expressed with the help of this thermostatic primitive function :

(1.1.20) $\qquad \eta(W) \,=\, -\Sigma\,(\rho\,,\, 1\,,\, \rho\, e)\,.$

Taking into account the relations (1.1.8) and (1.1.19), we easily differentiate the relation (1.1.20) and it comes

(1.1.21) $$ {\rm d}\eta(W) \,=\, \frac{1}{T}\Big(\mu - \frac{u^2}{2}\Big)\,{\rm d}\rho \,+\, \frac{u}{T}\,{\rm d}(\rho\, u) \,-\, \frac{1}{T}\,{\rm d}(\rho\, E)\,.$$

By comparison between (1.1.16) and (1.1.21) we have

(1.1.22) $$ \phi \,=\, \frac{1}{T}\Big(\,\mu - \frac{u^2}{2}\,,\, u\,,\, -1\Big)\,.$$

1.2 Linear and nonlinear waves.

• In this section, we construct particular solutions of the so-called Riemann problem. First recall that the Riemann problem consists in searching an entropy solution (*i.e.* a solution satisfying (1.1.15) in the sense of distributions, see *e.g.* Godlewski-Raviart [GR96] or Serre [Se96]) $W(x,\, t)$ $(x \,\in\, \mathbb{R},\; t \,>\, 0)$ of the following Cauchy problem :

(1.2.1) $$ \frac{\partial W}{\partial t} \,+\, \frac{\partial}{\partial x}\, F(W) \,=\, 0$$

(1.2.2) $$ W(x\,,\, 0) \,=\, \begin{cases} W_l\,, & x \,<\, 0 \\ W_r\,, & x \,>\, 0\,. \end{cases}$$

We first remark that conservation law (1.2.1) is **invariant under a change of the space-time scale**. Space-time transform T_λ parameterized by $\lambda > 0$, and defined by the relation $T_\lambda(x,\, t) = (\lambda\, x,\, \lambda\, t)$ can be applied on any (weak) solution $W(\bullet\,,\, \bullet)$ of system (1.2.1)-(1.2.2) and generates a new weak solution $T_\lambda\, W(x,\, t)$ defined by the condition $(T_\lambda\, W)(x,\, t) \equiv W(T_\lambda(x,\, t))\,.$ We remark also that the initial condition (1.2.2) is invariant under space dilatation, *i.e.*

(1.2.3) $\qquad W(\lambda\, x\,,\, 0) \;=\; W(x\,,\, 0)\;, \qquad \forall\, \lambda \,>\, 0\,.$

Then an hypothesis of **unicity** for weak solution of problem (1.2.1)-(1.2.2) shows that the solution $W(\bullet\,,\,\bullet)$ must be self-similar. This property can be expressed by the relation

(1.2.4) $\qquad W(\lambda\, x\,,\, \lambda\, t) \;=\; W(x\,,\, t)\;, \qquad \forall\, \lambda \,>\, 0\,.$

The relation (1.2.4) claims that solution $W(x,\,t)$ must be searched under a selfsimilar form, *i.e.* under the form of a function of the variable $\frac{x}{t}$ (see more details *e.g.* in the book of Landau and Lifchitz [LL54]) :

(1.2.5) $\qquad W(x\,,\, t) \;=\; U(\xi)\;, \qquad \xi \;=\; \dfrac{x}{t}\,.$

• As a first step, we consider regular solutions $\xi \longmapsto U(\xi)$ of the Riemann problem (1.2.1) (1.2.2). We introduce the representation (1.2.5) inside conservation law (1.2.1) and we obtain by this way :

(1.2.6) $\qquad {\rm d}F(U(\xi))\,\bullet\, \dfrac{{\rm d}U}{{\rm d}\xi} \;=\; \xi\, \dfrac{{\rm d}U}{{\rm d}\xi}\,.$

We deduce from (1.2.6) that we are necessarily in one of the two following cases : either vector $\frac{{\rm d}U}{{\rm d}\xi}$ is equal to zero or this vector is not equal to zero. In the first case, the solution is a constant state and in the second opportunity, the vector $\frac{{\rm d}U}{{\rm d}\xi}$ is necessarily equal to some eigenvector $R(W)$ of the jacobian matrix ${\rm d}F(W)$. In this second case, we have the classical relation between the jacobian matrix, eigenvector and eigenvalue $\lambda(W)$:

(1.2.7) $\qquad {\rm d}F(W)\,\bullet\, R(W) \;=\; \lambda(W)\, R(W)$

in the particular case where $W \,=\, U(\xi)$. By identification between the two relations (1.2.6) and (1.2.7), we deduce that the vectors $\frac{{\rm d}U}{{\rm d}\xi}$ and $R(U)$ are proportional and we deduce also :

(1.2.8) $\qquad \lambda\big(U(\xi)\big) \;=\; \xi\,.$

We have derived the conditions that characterize a so-called **rarefaction wave.**

• It is also possible to suppose that function $\xi \longmapsto U(\xi)$ admits some point of discontinuity at $\xi = \sigma$. It satisfies necessarily the Rankine-Hugoniot relations that link the jump of state $[\,U\,]$, the jump of the flux $[\,F(U)\,]$ and the celerity σ of the discontinuity profile :

(1.2.9) $\qquad \big[\,F(U)\,\big] \;=\; \sigma\,\big[\,U\,\big]\,.$

Recall that the Rankine-Hugoniot relations express that the discontinuous function $\xi \longmapsto U(\xi)$ is a weak solution of conservation (1.2.1) (see *e.g.* the book of Godlewski and Raviart [GR96]). By this way, we are deriving a so-called **shock wave**.

• We have observed in the previous section that the construction of a rarefaction wave is associated with some eigenvalue of the jacobian matrix ${\rm d}F(W)$. In the case of the Euler equations of gas dynamics, these eigenvalues are simply

computed with the help of the so-called nonconservative form of the equations that are obtained with the introduction of specific entropy s (see [LL54] for the proof) :

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + \rho \frac{\partial u}{\partial x} = 0 \tag{1.2.10}$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0 \tag{1.2.11}$$

$$\frac{\partial s}{\partial t} + u \frac{\partial s}{\partial x} = 0 \,. \tag{1.2.12}$$

Then we set :

$$Z(W) \equiv \big(\rho \,,\, u \,,\, s \big)^{\rm t} \tag{1.2.13}$$

$$B(Z) = \begin{pmatrix} u & \rho & 0 \\ \dfrac{c^2}{\rho} & u & \dfrac{1}{\rho} \dfrac{\partial p}{\partial s}(\rho,\, s) \\ 0 & 0 & u \end{pmatrix} \tag{1.2.14}$$

and the Euler equations that took the form (1.2.10)-(1.2.12) can be also written with the above matrix :

$$\frac{\partial Z}{\partial t} + B(Z) \bullet \frac{\partial Z}{\partial x} = 0 \,. \tag{1.2.15}$$

Celerity c for sound waves used in formula (1.2.14) is defined by

$$c^2 = \frac{\partial p}{\partial \rho}(\rho,\, s) \tag{1.2.16}$$

and for a perfect polytropic gas with ratio γ of specific heats, we have :

$$c = \sqrt{\frac{\gamma\, p}{\rho}} \,. \tag{1.2.17}$$

• If we wish to diagonalize the matrix $\mathrm{d}F(W)$, it is sufficient to make this job for matrix $B(Z)$ because they are **conjugate** as we show in the following. We introduce the vector variable Z inside equation (1.2.1) :

$$\mathrm{d}W(Z) \bullet \frac{\partial Z}{\partial t} + \mathrm{d}F(W) \bullet \mathrm{d}W(Z) \bullet \frac{\partial Z}{\partial x} = 0 \tag{1.2.18}$$

and by comparison with relation (1.2.15) we have necessarily :

$$B(Z) = (\mathrm{d}W(Z))^{-1} \bullet \mathrm{d}F(W) \bullet \mathrm{d}W(Z) \,. \tag{1.2.19}$$

This conjugation relation shows that $\widetilde{R}(Z) \equiv (\mathrm{d}W(Z))^{-1} \bullet R(W(Z))$ is an eigenvector for the matrix $B(Z(W))$ with the eigenvalue $\lambda(W)$ if $R(W)$ is some eigenvector of the jacobian matrix $\mathrm{d}F(W)$ that satisfies the relation (1.2.7). The diagonalization of matrix $B(Z)$ is straightforward. We find

$$\lambda_1 = u - c \quad < \quad \lambda_2 = u \quad < \quad \lambda_3 = u + c \tag{1.2.20}$$

with associated eigenvectors given by the following formulae :

$$\widetilde{R}_1(Z) \;=\; \begin{pmatrix} \rho \\ -c \\ 0 \end{pmatrix} \,, \quad \widetilde{R}_2(Z) \;=\; \begin{pmatrix} \frac{\partial p}{\partial s} \\ 0 \\ -c^2 \end{pmatrix} \,, \quad \widetilde{R}_3(Z) \;=\; \begin{pmatrix} \rho \\ c \\ 0 \end{pmatrix} . \tag{1.2.21}$$

We remark that the derivation of scalar field λ_1 (respectively λ_3) inside direction $\widetilde{R}_1$ (respectively $\widetilde{R}_3$) is never null

$$\mathrm{d}\lambda_1(W(Z)) \bullet \widetilde{R}_1(W) \;\neq\; 0 \,, \qquad \mathrm{d}\lambda_3(W(Z)) \bullet \widetilde{R}_3(W) \;\neq\; 0 \,, \qquad \forall\, W \tag{1.2.22}$$

whereas we have the opposite situation for the second eigenvalue λ_2 :

$$\mathrm{d}\lambda_2(W(Z)) \bullet \widetilde{R}_2(W) \;=\; 0 \,, \qquad \forall\, W\,. \tag{1.2.23}$$

For this reason, we will say that the eigenvalues λ_1 and λ_3 define **genuinely nonlinear** fields whereas the eigenvalue $\lambda_2 = u$ defines a **linearly degenerate** field. Rarefaction and shock waves are always associated with genuinely nonlinear fields and in what follows, we distinguish between 1-rarefaction wave and 3-rarefaction wave.

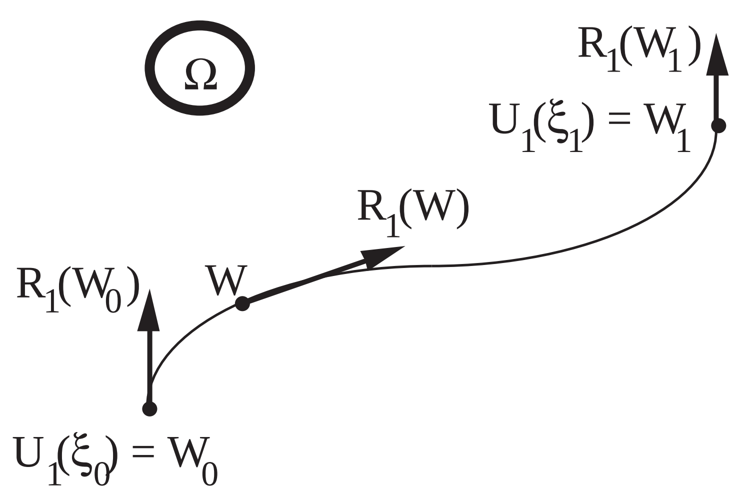

Figure 1.1 *Rarefaction wave associated with eigenvalue* $\lambda_1 = u - c$ *in the space of states ; the curve is everywhere tangent to eigenvector* $R_1(W)$.

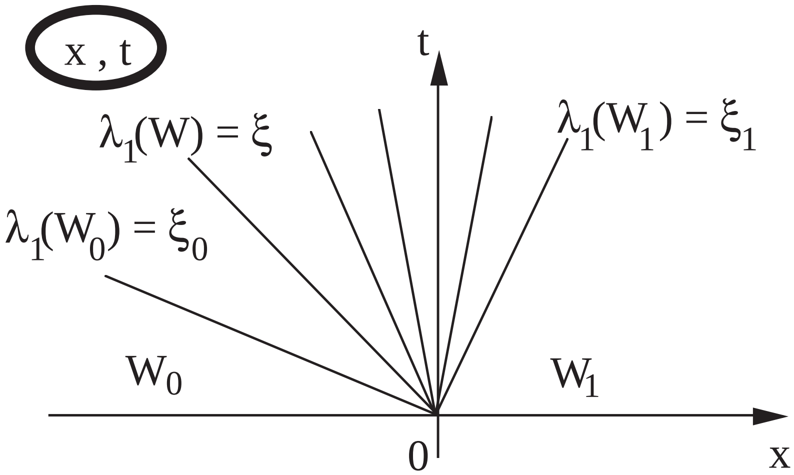

Figure 1.2 *Rarefaction wave* $U_1(\xi)$ *associated with eigenvalue* $\lambda_1 = u - c$ *inside the space-time plane.*

• A 1-rarefaction wave is a function $\xi \longmapsto U_1(\xi)$ that satisfies

(1.2.24) $\quad \dfrac{\mathrm{d}}{\mathrm{d}\xi}\big(U_1(\xi)\big)$ proportional to $R_1\big(U_1(\xi)\big)$

and we have an analogous definition for a 3-wave $\xi \longmapsto U_3(\xi)$: $\frac{\mathrm{d}}{\mathrm{d}\xi}\big(U_3(\xi)\big)$ is proportional to $R_3\big(U_3(\xi)\big)$. Then if we integrate the vector field $\widetilde{R}_1(Z)$ *i.e.* if we solve the ordinary differential equation $\frac{\mathrm{d}Z}{\mathrm{d}\xi} = \widetilde{R}_1(Z(\xi))$, we find a function $\xi \longmapsto Z(U_1(\xi))$ that defines a 1-rarefaction wave, say for variable ξ contained between two limiting values ξ_0 and ξ_1. We can draw the solution of differential equation (1.2.24) inside the space of states W ; we find a curve $\xi \longmapsto U_1(\xi)$ that satisfies equation (1.2.24) and the initial condition

(1.2.25) $\quad U_1(\xi_0) \,=\, W_0 \,.$

Moreover this curve is defined up to $\xi = \xi_1$ where the final state W_1 is achieved (see Figure 1.1)

(1.2.26) $\quad U_1(\xi_1) \,=\, W_1 \,.$

In space-time $(x,\, t)$, we observe that we have to consider three rates of flow for celerities $\xi = \frac{x}{t}$:

(1.2.27) $\quad U_1(\xi) \,=\, W_0 \quad$ for $\quad \xi \,\leq\, \xi_0 \quad$ (constant state)
(1.2.28) $\quad U_1(\xi)$ variable $\quad$ for $\quad \xi_0 \,\leq\, \xi \,\leq\, \xi_1 \quad$ (rarefaction wave)
(1.2.29) $\quad U_1(\xi) \,=\, W_1 \quad$ for $\quad \xi \,\geq\, \xi_1 \quad$ (constant state)

as proposed at Figure 1.2. Notice that relation (1.2.8) imposes a particular value ξ_0 for celerity :

(1.2.30) $\quad \xi_0 \,=\, u(W_0) - c(W_0)$

and an analogous value for end-point ξ_1. It is simple to show (see *e.g.* Smoller [Sm83] or Godlewski-Raviart [GR96]) that inequality

(1.2.31) $\xi_0 \,\leq\, \xi_1$

is necessary if W_1 is linked to W_0 through a -1- or a 3-rarefaction wave. In the particular case of a 3-rarefaction wave, relations (1.2.27) to (1.2.29) are still correct, but relation (1.2.30) must be replaced by

(1.2.32) $\xi_0 \,=\, u(W_0) + c(W_0)$

and we have also an analogous relation for ξ_1.

1.3 Riemann invariants and rarefaction waves.

• The practical computation of the curve $\xi \longmapsto U_1(\xi)$ satisfying the relations (1.2.24) and (1.2.25) uses the notion of Riemann invariant. By definition, a 1-Riemann invariant (respectively a 3-Riemann invariant) is a function $W \longmapsto \beta^1(W)$ (respectively $W \longmapsto \beta^3(W)$) that is constant along the curves of 1-rarefactions (respectively along the curves of 3-rarefactions) and satisfies by definition :

(1.3.1) $\mathrm{d}\beta^1(W) \bullet R_1(W) \,=\, 0 \qquad \forall\, W$

(respectively $\mathrm{d}\beta^3(W) \bullet R_3(W) \,=\, 0$ for each state W). If we express this relation in terms of nonconservative variables Z, we set $\widetilde{\beta}(Z) \equiv \beta\big(W(Z)\big)$ and we have : $\mathrm{d}\widetilde{\beta}(Z) \bullet \widetilde{R}(Z) \,=\, \mathrm{d}\beta(W(Z)) \bullet \mathrm{d}W(Z) \bullet \big(\mathrm{d}W(Z)\big)^{-1} \bullet R(W(Z)) \,=\, 0$ if $\beta(\bullet)$ is a Riemann invariant for the field associated with eigenvector $R(\bullet)$. We deduce from the previous calculus the relation :

(1.3.2) $\mathrm{d}\beta^1(W(Z)) \bullet \widetilde{R}_1(Z) \,=\, 0 \qquad \forall\, Z\,.$

Taking into account the particular form (1.2.21) of the vector $\widetilde{R}_1(Z)$, the two following functions

(1.3.3) $\beta^1_1(W) \,=\, s$

(1.3.4) $\beta^1_2(W) \,=\, u + \displaystyle\int_{\rho_0}^{\rho} \frac{c(\theta,\, s)}{\theta}\,\mathrm{d}\theta \,=\, u + \frac{2\,c}{\gamma - 1}$

are particular 1-Riemann invariants ; they satisfy together the relation (1.3.2). The expression of the 3-Riemann invariants is obtained by an analogous way :

(1.3.5) $\beta^3_1(W) \,=\, s$

(1.3.6) $\beta^3_2(W) \,=\, u - \displaystyle\int_{\rho_0}^{\rho} \frac{c(\theta,\, s)}{\theta}\,\mathrm{d}\theta \,=\, u - \frac{2\,c}{\gamma - 1}\,.$

The states W on a 1-rarefaction wave issued from state W_0 are explicited by using the relation (1.3.1) for the Riemann invariants proposed at relations (1.3.3) and (1.3.4). This fact express that on a 1-rarefaction wave, the two associated Riemann invariants are constant ; we have

(1.3.7) $s \,=\, s_0$

(1.3.8) $$u + \frac{2\,c}{\gamma-1} \;=\; u_0 + \frac{2\,c_0}{\gamma-1}$$

(1.3.9) $$\xi \;=\; u - c\,, \qquad \xi \,\geq\, u_0 - c_0\,.$$

• We detail now the particular algebraic form that takes the description of the link beween a state W and its initial datum W_0 through a 1-rarefaction wave inside a perfect polytropic gas. We first set (we refer for the details to the book of Courant and Friedrichs [CF48]) that inside a 1-rarefaction wave, the pressure p is a decreasing function of velocity :

(1.3.10) $p \,\leq\, p_0$, $u \,\geq\, u_0$, W issued from W_0 *via* a 1-rarefaction

and due to the expression of the entropy s for a polytropic perfect gas,

(1.3.11) s is a function of variable $\dfrac{p}{\rho^\gamma}$

it comes, taking into account the relations (1.2.17), (1.3.6) and (1.3.7),

(1.3.12) $$u - u_0 + \frac{\sqrt{1-\mu^4}}{\mu^2}\,\frac{p_0^{\frac{1}{2\gamma}}}{\sqrt{\rho_0}}\left(p^{\frac{\gamma-1}{2\gamma}} - p_0^{\frac{\gamma-1}{2\gamma}}\right) \;=\; 0 \qquad \text{(1-rarefaction)}$$

parameterized by the non-dimensional coefficient $\mu > 0$ [note that this parameter μ has nothing to do with the chemical potential, even if the same letter is used !] defined by

(1.3.13) $$\mu^2 \;=\; \frac{\gamma-1}{\gamma+1}\,.$$

In the plane (u, p) composed by velocity and pressure, the graphic representation of the 1-rarefaction (relation (1.3.12) under condition (1.3.10)) is proposed on Figure 1.3.

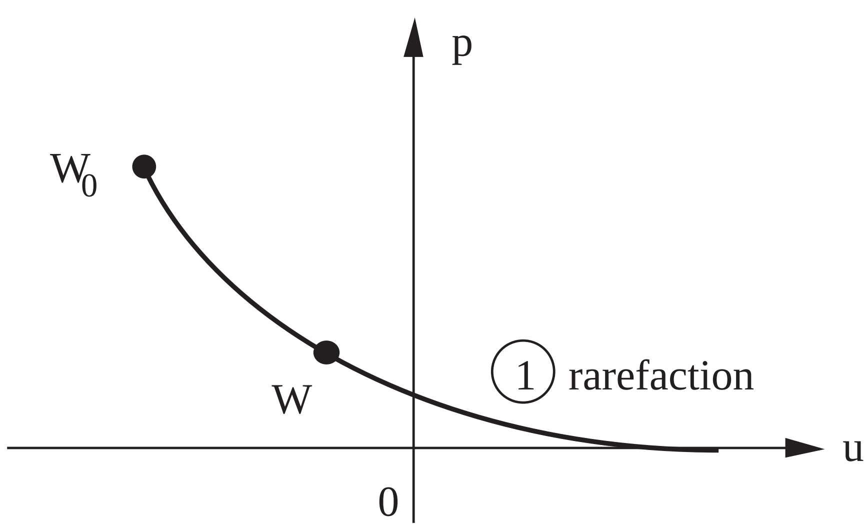

Figure 1.3 *1-rarefaction wave linking state W_0 with state W in the plane of velocity and pressure.*

• For a 3-rarefaction wave, we look for (due to reasons that will be explained in what follows) an upstream state W linked to a **downstream** state W_0. We obtain

(1.3.14) $$s \,=\, s_0$$

(1.3.15) $$u - \frac{2\,c}{\gamma-1} \,=\, u_0 - \frac{2\,c_0}{\gamma-1}$$

(1.3.16) $$\xi \,=\, u + c\,, \qquad \xi \leq u_0 + c_0\,.$$

It is also easy to prove that in a 3-rarefaction, velocity is a nondecreasing function of pressure, *i.e.* :

(1.3.17) $$p \,\leq\, p_0\,, \qquad u \,\leq\, u_0\,, \qquad W_0 \text{ issued from } W \textit{ via} \text{ a 3-rarefaction.}$$

A computation analogous to the one presented for 1-rarefactions shows

(1.3.18) $$u - u_0 - \frac{\sqrt{1-\mu^4}}{\mu^2}\,\frac{p_0^{\frac{1}{2\gamma}}}{\sqrt{\rho_0}}\left(p^{\frac{\gamma-1}{2\gamma}} - p_0^{\frac{\gamma-1}{2\gamma}}\right) \,=\, 0 \qquad \text{(3-rarefaction).}$$

The comparison between relations (1.3.12) and (1.3.18) induces us to set

(1.3.19) $$\psi(p\,;\,\rho_0\,,\,p_0\,;\,\gamma) \,\equiv\, \frac{\sqrt{1-\mu^4}}{\mu^2}\,\frac{p_0^{\frac{1}{2\gamma}}}{\sqrt{\rho_0}}\left(p^{\frac{\gamma-1}{2\gamma}} - p_0^{\frac{\gamma-1}{2\gamma}}\right).$$

The graph in the plane $(u,\,p)$ of the curve of 3-rarefaction (equation (1.3.18) under the constraint (1.3.17)) is presented on Figure 1.4.

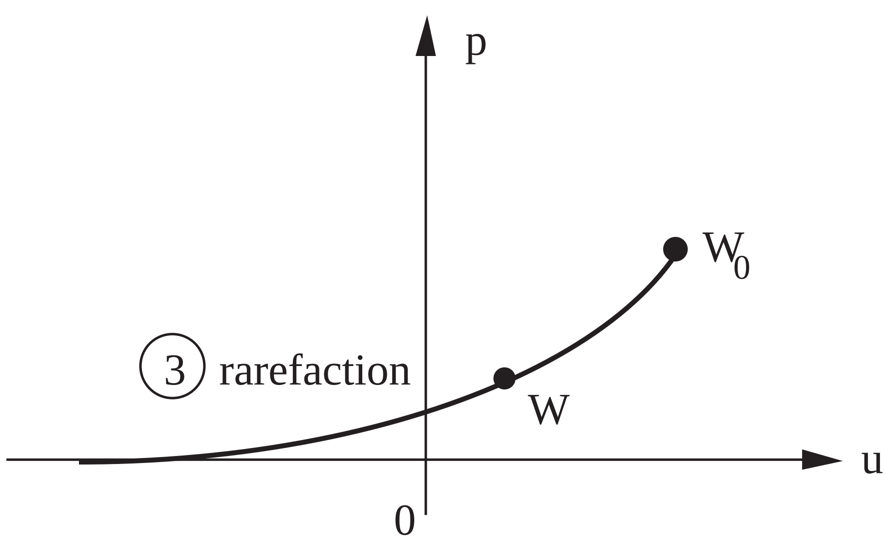

Figure 1.4 *Curve in the plane of velocity and pressure showing the set of states W such that the 3-rarefaction wave that begins at state W ends at the particular state W_0.*

• It can be usefull to precise the (varying) state inside a 1-rarefaction wave as a function of celerity $\xi = \frac{x}{t}$. Taking into account the relation (1.2.8),

we have on one hand :

(1.3.20) $\quad u - c = \xi$

and on the other hand taking into account the relation (1.3.8), it comes easily, for $u_l - c_l < \xi < u_1 - c_1$:

(1.3.21) $\quad u = \dfrac{\gamma-1}{\gamma+1} u_l + \dfrac{2}{\gamma+1} c_l + \dfrac{2}{\gamma+1} \xi$

(1.3.22) $\quad c = \dfrac{\gamma-1}{\gamma+1} u_l + \dfrac{2}{\gamma+1} c_l - \dfrac{\gamma-1}{\gamma+1} \xi \,.$

The elimination of pressure among the relations (1.2.17) and (1.3.7) allows the evaluation of density. It is the same set of operations for a 3-rarefaction wave. The relation (1.2.8) gives the expression of the celerity of the 3-wave as a function of the data :

(1.3.23) $\quad u + c = \xi$

and the Riemann invariant (1.3.15) allows the evaluation of velocity u and sound celerity c as a function of the data :

(1.3.24) $\quad u = \dfrac{\gamma-1}{\gamma+1} u_r - \dfrac{2}{\gamma+1} c_r + \dfrac{2}{\gamma+1} \xi$

(1.3.25) $\quad c = -\dfrac{\gamma-1}{\gamma+1} u_r + \dfrac{2}{\gamma+1} c_r + \dfrac{\gamma-1}{\gamma+1} \xi$

and under the conditions $u_2 + c_2 < \xi < u_r + c_r$.

1.4 Rankine-Hugoniot jump relations and shock waves.

• We consider now two particular states W_0 and W linked together *via* the Rankine-Hugoniot jump relations (1.2.9) and the associated entropy inequality. We first remark that the Galilean invariance of the equations of gas dynamics allows us to think the Physics inside the reference frame with a velocity exactly equal to the celerity σ of the discontinuity. Then the jump equations (1.2.9) are written again under the more detailed form

(1.4.1) $\quad \big[\rho\,(u-\sigma) \big] = 0$

(1.4.2) $\quad \big[\rho\,(u-\sigma)^2 + p \big] = 0$

(1.4.3) $\quad \Big[\rho\,(u-\sigma) \Big(e + \dfrac{(u-\sigma)^2}{2} \Big) + p\,(u-\sigma) \Big] = 0$

as it is also easy to derive directly. We consider the classical expression for mass flux that cross the shock wave :

(1.4.4) $\quad m = \rho\,(u-\sigma)\,.$

The associated conditions of increasing physical entropy through a shock wave (see *e.g.* [CF48] or [LL54]) claim that we have

(1.4.5) $\quad m > 0 \quad$ through a 1-shock

(1.4.6) $\quad m < 0 \quad$ through a 3-shock.

The numbering of shock waves can be explained by an argument of continuity as follows. If the jumps inside relations (1.4.1) to (1.4.3) are weak enough, it is possible to show (see details in [Sm83] or [GR96] for example) that the celerity σ of the discontinuity relative to both states W_0 and W has a limit value equal to the common value $u-c$ for a 1-shock and $u+c$ for a 3-shock wave. The particular case where the mass flux m is equal to zero will be considered afterwards and corresponds to a slip surface or **contact discontinuity**.

• We detail the algebra that is necessary in order to express that the state W is obtained from an upstream state W_0 through a 1-shock wave with celerity σ. We first remark that the entropy condition implies the following family of inequalities :

(1.4.7) $$\rho > \rho_0 \,, \qquad p > p_0 \,, \qquad u-c < \sigma < u_0 - c_0 \,, \qquad s > s_0$$

when W is issued from W_0 through a 1-shock wave. We denote also by h and τ the specific enthalpy and the specific volume

(1.4.8) $$h = e + \frac{p}{\rho}$$

(1.4.9) $$\tau = \frac{1}{\rho} \,.$$

From relations (1.4.1) to (1.4.3) we deduce, taking into account (1.4.4), (1.4.8) and (1.4.9) :

(1.4.10) $$\big[\, u \,\big] = m \big[\, \tau \,\big]$$

(1.4.11) $$m^2 \big[\, \tau \,\big] + \big[\, p \,\big] = 0$$

(1.4.12) $$\big[\, h \,\big] + \frac{m^2}{2} \big[\, \tau^2 \,\big] = 0 \,.$$

In the particular case of a polytropic perfect gas, we eliminate the mass flow m from relations (1.4.11) and (1.4.12) and we express the enthalpy as a function of pressure and specific volume, *i.e.*

(1.4.13) $$h = \frac{\gamma}{\gamma-1} \, p \, \tau \,.$$

After some lines of elementary algebra we express the specific volume τ downstream to the shock as a function of upstream data, downstream pressure and variable μ introduced at relation (1.3.13). We obtain :

(1.4.14) $$\tau = \frac{p_0 + \mu^2 \, p}{p + \mu^2 \, p_0} \, \tau_0 \,.$$

We report this particular expression inside relation (1.4.11) and obtain by this way the square of the mass flux across the shock :

(1.4.15) $$m^2 = \frac{p + \mu^2 \, p_0}{(1-\mu^2) \, \tau_0} \,, \qquad W \text{ issued from } W_0 \text{ by a 1-shock.}$$

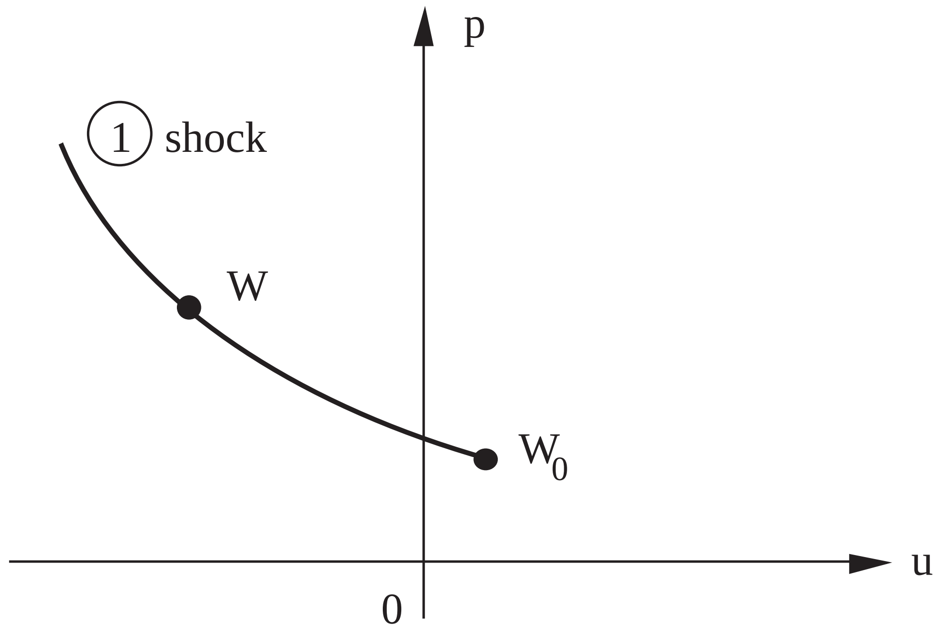

Figure 1.5 *Curve showing (in the velocity-pressure plane) the set of states W issued from the particular state W_0 through a 1-shock wave.*

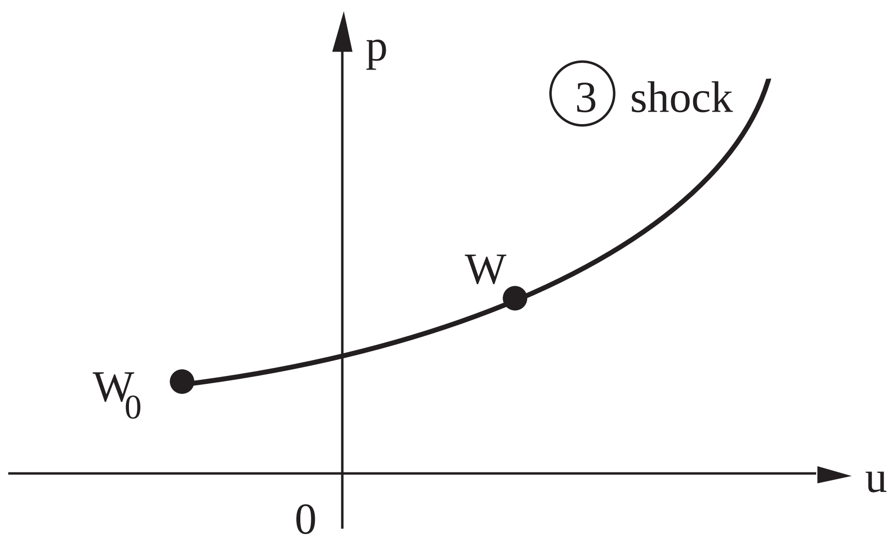

Figure 1.6 *Set of states W such that the 3-shock wave links state W to the particular state W_0.*

• For a 1-shock wave, the relation (1.4.2) joined with (1.4.10) allows us to precise the jump of velocity as a function of pressure and upstream state :

$$u - u_0 + \sqrt{\frac{1-\mu^2}{\rho_0 \, (p + \mu^2 \, p_0)}} \, (p - p_0) \; = \; 0 \qquad (1.4.16)$$

when W is issued from W_0 through a 1-shock wave. In the plane of velocity-pressure variables, we can propose (see Figure 1.5) the curve characterized by the equation (1.4.16) and the inequalities (1.4.7).

• For a 3-shock wave, the relations (1.4.10) to (1.4.13) are still valid. We do not consider in what follows a state W issued from W_0 *via* a 3-shock wave but we reverse the roles and consider the set of states W that are **upstream** to the particular state W_0 through a 3-shock wave. We reverse the zero index in previous formulae. With these particular hypotheses concerning our new notations, a simple calculus shows that relation (1.4.15) remains still correct and we have

$$m^2 \;=\; \frac{p \,+\, \mu^2\, p_0}{(1-\mu^2)\,\tau_0}\,, \qquad W_0 \text{ issued from } W \text{ by a 3-shock.} \tag{1.4.17}$$

The algebraic relation that express the jump of velocity as a function of downstream state W_0 and upstream pressure takes now the form

$$u \,-\, u_0 \,-\, \sqrt{\frac{1-\mu^2}{\rho_0\,(p+\mu^2\,p_0)}}\,(p-p_0) \;=\; 0 \tag{1.4.18}$$

when W_0 is issued from W through a 3-shock wave. Taking into account these new notations, the inequalities for entropy condition can be written as

$$\rho_0 \,<\, \rho\,, \qquad p_0 \,<\, p\,, \qquad u_0 + c_0 \,<\, \sigma \,<\, u + c\,, \qquad s_0 \,<\, s \tag{1.4.19}$$

if W_0 is issued from W through a 3-shock wave. A 3-shock wave is graphically represented represented at Figure 1.6. By comparison between the relations (1.4.16) and (1.4.18), it is natural to define

$$\varphi\,(p\,;\,\rho_0\,,\,p_0\,;\,\gamma) \;\equiv\; \sqrt{\frac{1-\mu^2}{\rho_0\,(p+\mu^2\,p_0)}}\,(p-p_0) \tag{1.4.20}$$

where parameters μ and γ are linked by relation (1.3.13).

1.5 Contact discontinuities.

• When we have studied the rarefaction waves, we have introduced the notion of linearly degenerated field at relation (1.2.23) and we have observed that the second characteristic field of the Euler equations of gas dynamics is effectively linearly degenerated. This fact means that the second eigenvalue λ_2 is also a Riemann invariant :

$$\beta_1^2(W) \;=\; u\,. \tag{1.5.1}$$

Taking into account the particular expression (1.2.21) of eigenvectors, it is easy to see that the pressure is also a 2-Riemann invariant independent from the first one :

$$\beta_2^2(W) \;=\; p\,. \tag{1.5.2}$$

We can search a selfsimilar regular wave $\xi \longmapsto U(\xi)$ solution of the differential equation that expresses proportionality between $\frac{\mathrm{d}U}{\mathrm{d}\xi}$ and $R_2(U)$, but in that case the necessary condition (1.2.8) implies that variable ξ is not allowed to get any variation ! We have in fact by derivation of relation (1.2.8)

$$\mathrm{d}\lambda_2(U) \bullet \frac{\mathrm{d}U}{\mathrm{d}\xi} \;=\; 1 \tag{1.5.3}$$

whereas this last quantity is identically equal to zero (relation (1.2.23)) if we consider the hypothesis of linearly degeneration of second characteristic field $\lambda_2 \equiv u$.

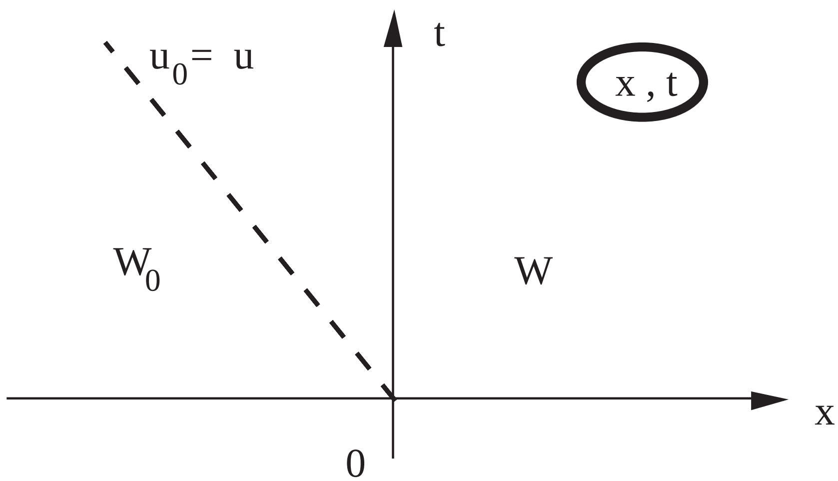

Figure 1.7 *Contact discontinuity between the two states W_0 and W.*

• We remark also that if the function $\xi \longmapsto U(\xi)$ is equal to some integral curve of vector field R_2, *i.e.*

$$\frac{\mathrm{d}U}{\mathrm{d}\xi} \;=\; R_2(U) \tag{1.5.4}$$

then we have between states W_0 and W the following calculus :

$$\begin{aligned} F(W) - F(W_0) \;&=\; \int_{\xi_0}^{\xi} \mathrm{d}F(U(\eta)) \bullet \frac{\mathrm{d}U}{\mathrm{d}\eta}\,\mathrm{d}\eta \;=\; \int_{\xi_0}^{\xi} \mathrm{d}F(U(\eta)) \bullet R_2(U(\eta))\,\mathrm{d}\eta \\ &=\; \int_{\xi_0}^{\xi} \lambda_2(U(\eta))\, R_2(U(\eta))\,\mathrm{d}\eta \\ &=\; \lambda_2 \int_{\xi_0}^{\xi} R_2(U(\eta))\,\mathrm{d}\eta \;=\; \lambda_2\,(W - W_0)\,. \end{aligned}$$

The fourth step in the previous relations is a consequence of the linear degenerescence of velocity u which is a constant along the 2-curve defined at relation

(1.5.4). We have just shown that we have a Rankine-Hugoniot jump relation between states W_0 and W :

(1.5.5) $\quad \big[F(W) \big] \;=\; \lambda_2 \big[W \big] \,.$

In the space of states, the rarefaction waves defined by the differential relation (1.5.4) and the shock curves that we founded at relation (1.5.5) are identical ! The second characteristic field is linearly degenerate and defines a curve in the space of states that joins the two states W_0 and W according to the relations

(1.5.6) $\quad u \;=\; u_0$

(1.5.7) $\quad p \;=\; p_0 \,.$

In space-time variables, these two states are linked through a contact discontinuity with celerity σ equal to λ_2 due to relation (1.5.5) :

(1.5.8) $\quad \sigma \;=\; u \;=\; u_0 \,.$

Such a wave is called **contact discontinuity** or **slip line**. We have

(1.5.9) $\quad U_2(\xi) \;=\; \begin{cases} W_0 \,, & \xi \,<\, u_0 \;=\; \sigma \\ W \,, & \xi \,>\, u_0 \;=\; \sigma \,. \end{cases}$

A representation in space-time of relation (1.5.9) is proposed at Figure 1.7. In the velocity-pressure plane, the (nonlinear !) projections of states W_0 and W coincide, as previously established in relations (1.5.6) and (1.5.7).

1.6 Practical solution of the Riemann problem.

• We propose to solve the Riemann problem (1.2.1)-(1.2.2) between two states W_l and W_r, and we remark that the general theory proposed by Lax (see *e.g.* [Lax73]) can be applied for gas dynamics. We search two intermediate states W_1 and W_2 such that

(1.6.1) $\quad W_1$ is issued from state W_l by a 1-wave

(1.6.2) $\quad W_2$ is issued from state W_1 by a 2-wave

(1.6.3) $\quad W_r$ is issued from state W_2 by a 3-wave.

As a first step, we restrict ourselves to the search of velocity and pressure which is common to both states W_1 and W_2 due to the fact that the 2-wave is a contact discontinuity :

(1.6.4) $\quad u_1 \;=\; u_2 \;=\; u^*$

(1.6.5) $\quad p_1 \;=\; p_2 \;=\; p^* \,.$

• The relation (1.6.1) means that state W_1 is issued from state W_l through a 1-rarefaction wave (relation (1.3.10)) or a 1-shock wave (inequalities (1.4.7)). We have in consequence :

(1.6.6) $\quad \begin{cases} u_1 \,-\, u_l \,+\, \psi(p_1 \,;\, \rho_l \,,\, p_l \,;\, \gamma) \;=\; 0 \,, & p_1 \,<\, p_l \\ u_1 \,-\, u_l \,+\, \varphi(p_1 \,;\, \rho_l \,,\, p_l \,;\, \gamma) \;=\; 0 \,, & p_1 \,>\, p_l \,. \end{cases}$

In a similar way, we remark that relation (1.6.3) expresses that state W_2 is the upstream state if a 3-wave whose downstream state is exactly the datum W_r.

We can use a 3-rarefaction wave between W_2 and W_r (relation (1.3.17)) or a 3-shock wave (relation (1.4.19)). It comes :

$$(1.6.7) \qquad \begin{cases} u_2 \,-\, u_r \,-\, \psi(p_2\,;\,\rho_r\,,\,p_r\,;\,\gamma) \;=\; 0\,, & p_2 \,<\, p_r \\ u_2 \,-\, u_r \,-\, \varphi(p_2\,;\,\rho_r\,,\,p_r\,;\,\gamma) \;=\; 0\,, & p_2 \,>\, p_r\,. \end{cases}$$

It is sufficient to write the equation (1.6.4) that links velocities u_1 and u_2 (*i.e.* $u_1 = u_2 = u^*$) under the condition (1.6.5) related to the common pressures p_1 and p_2 ($p_1 = p_2 = p^*$) to set completely the problem (see also Figure 1.8).

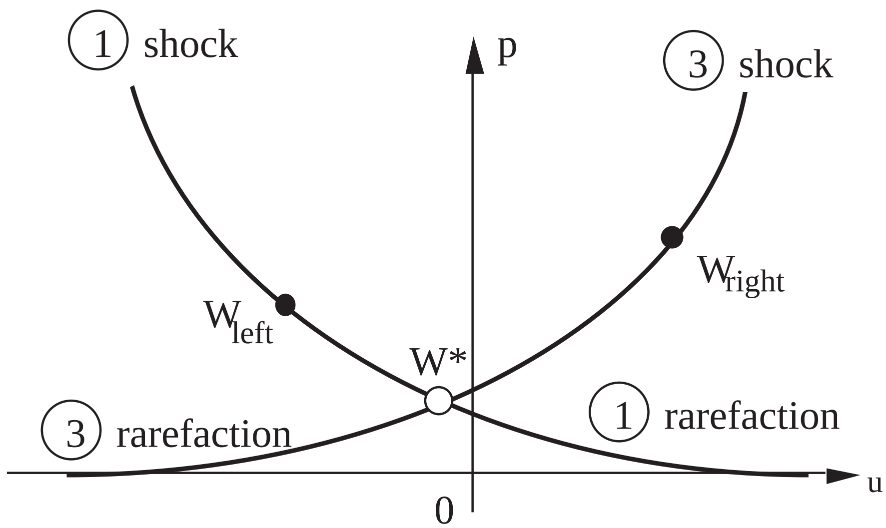

Figure 1.8 *Solution of the Riemann problem* $R(W_{\text{left}}\,,\,W_{\text{right}})$ *in the velocity-pressure plane.*

• The numerical resolution of problem (1.6.4)-(1.6.7) can be done by Newton iterations indexed by some integer k and presented on Figure 1.9. Starting from a given pressure p_k, we easily compute with relations (1.6.6) and (1.6.7) velocities $u_{1,\,k}$ and $u_{3,\,k}$ respectively associated with the 1-wave and the 3-wave. We evaluate the locus of intersection of the two tangent lines issued from the two corresponding velocities in order to define a new value p_{k+1} for pressure at iteration $k+1$. The initialization of the algorithm can be obtained by the intersection of the two rarefaction waves, *i.e.* by solving the following system with unknowns $(u_O,\,p_O)$:

$$(1.6.8) \qquad \begin{cases} u_O \,-\, u_l \,+\, \psi(p_O\,;\,\rho_l\,,\,p_l\,;\,\gamma) \;=\; 0 \\ u_O \,-\, u_r \,+\, \psi(p_O\,;\,\rho_r\,,\,p_r\,;\,\gamma) \;=\; 0\,. \end{cases}$$

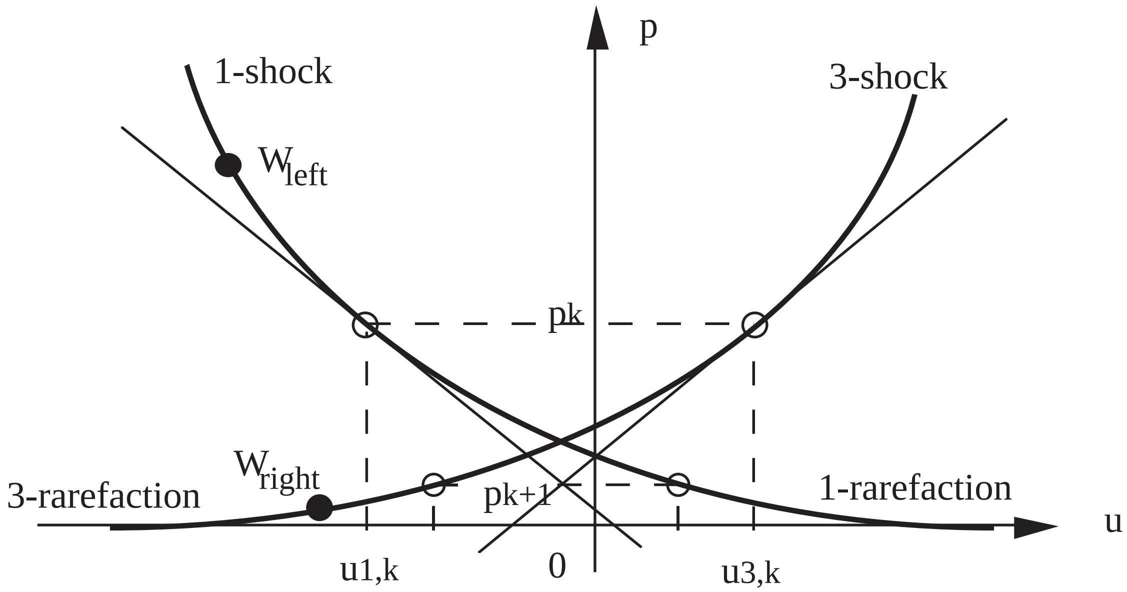

Figure 1.9 *Newton iterations for the resolution of the Riemann problem in the plane of velocity and pressure.*

Note that in relations (1.6.8), the index "O" stands for "Osher" because solving equations (1.6.8) is essentially what is to be done for the computation of the Osher [Os81] flux decomposition (see *e.g.* [Du87]). This system of equations, even if it is a nonlinear one, can be solved exactly with some explicit algebra and pressure p_O is finally evaluated thanks to the relation :

$$(1.6.9) \qquad p_O^{\frac{\gamma-1}{2\gamma}} \;=\; \frac{\frac{(\gamma-1)}{2}\,(u_l - u_r) \,+\, c_l + c_r}{c_l\left(\frac{1}{p_l}\right)^{\frac{\gamma-1}{2\gamma}} \,+\, c_r\left(\frac{1}{p_r}\right)^{\frac{\gamma-1}{2\gamma}}}\,,$$

where sound celerities c_l and c_r are evaluated with relation (1.2.17). We remark that relation (1.6.9) defines effectively a **positive** pressure if the following relation of **non vacuum** appearance is satisfied :

$$(1.6.10) \qquad u_r \,-\, u_l \;\leq\; \frac{2}{\gamma-1}\,(c_l + c_r)\,.$$

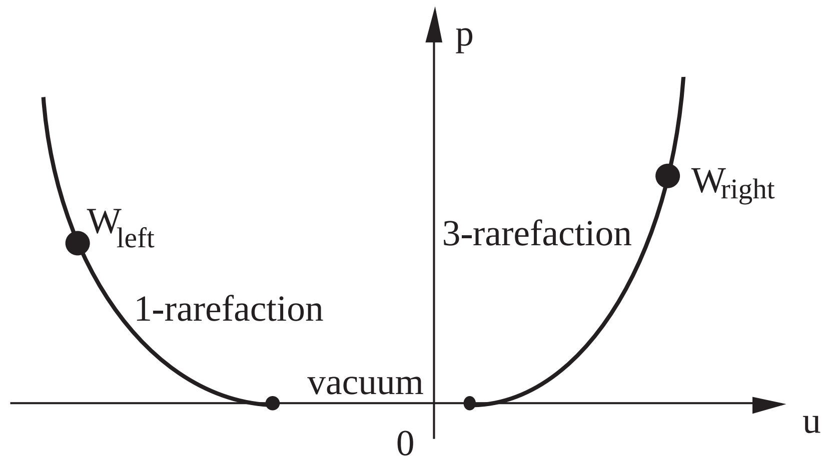

Figure 1.10 *Apparition of cavitation phenomenon in the velocity-pressure plane.*

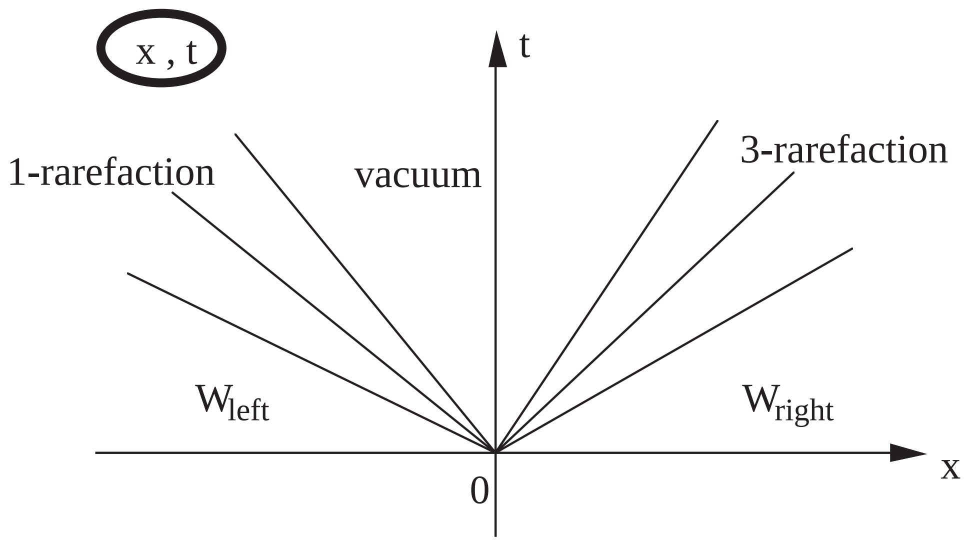

Figure 1.11 *Apparition of cavitation phenomenon. Solution of the Riemann problem in the space-time plane.*

• If in contrary relation (1.6.10) does not hold, the two curves associated with both rarefaction waves do not intersect in the velocity-pressure plane. A vacuum appears (see Figures 1.10 and 1.11). The solution of the Riemann problem is no longer mathematically well defined in the sense of Lax and contains a zone without any matter defined by celerities ξ such that

(1.6.11) $$u_l + \frac{2\,c_l}{\gamma-1} \;\leq\; \xi \;\leq\; u_r - \frac{2\,c_r}{\gamma-1}\,.$$

For such celerities, the pressure and the density are null whereas velocity is not defined. This cavitation process remains an exception but can be still completely solved as we have just seen.

• When relation (1.6.10) is satisfied, the Newton algorithm illustrated at Figure 1.9 is convergent towards a pair (u^*, p^*) composed by the common velocity $u^* \in \mathbb{R}$ and pressure $p^* > 0$ of the two intermediate states W_1 and W_2. The calculus of the density of these two intermediate states depends on the choice of the wave effectively used for the resolution of the Riemann problem. If pressure p^* is less or equal to the left pressure p_l, the 1-wave is a rarefaction wave ; then the entropy remains constant and we have

$$\rho_1 = \Big(\frac{p^*}{p_l}\Big)^{\frac{1}{\gamma}} \rho_l \ , \qquad p^* < p_l \, . \tag{1.6.12}$$

On the opposite case, we use a 1-shock wave and taking into account the relation (1.4.9), we get finally

$$\rho_1 = \frac{p^* + \mu^2 \, p_l}{p_l + \mu^2 \, p^*} \, \rho_l \ , \qquad p^* > p_l \, . \tag{1.6.13}$$

The celerity σ_1 of the shock wave can be explicited :

$$\sigma_1 = u_l - \sqrt{\frac{p^* + \mu^2 \, p_l}{(1-\mu^2)\, \rho_l}} \qquad p^* > p_l \, . \tag{1.6.14}$$

• For the 3-wave, there is an analogous discussion that conducts finally to the following relations :

$$\rho_3 = \Big(\frac{p^*}{p_r}\Big)^{\frac{1}{\gamma}} \rho_r \ , \qquad p^* < p_r \tag{1.6.15}$$

$$\rho_3 = \frac{p^* + \mu^2 \, p_r}{p_r + \mu^2 \, p^*} \, \rho_r \ , \qquad p^* > p_r \tag{1.6.16}$$

$$\sigma_3 = u_r + \sqrt{\frac{p^* + \mu^2 \, p_r}{(1-\mu^2)\, \rho_r}} \qquad p^* > p_r \, . \tag{1.6.17}$$

The qualitative comportment in space-time of the solution of a Riemann problem for gas dynamics is illustrated on Figure 1.12.

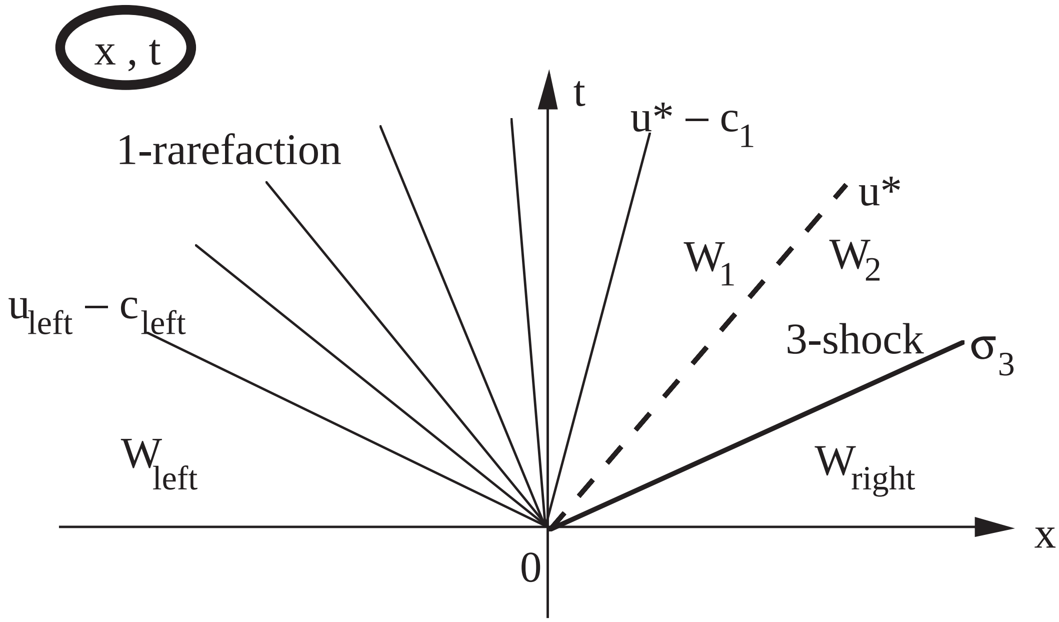

Figure 1.12 *Solution of the Riemann problem* $R(W_{\rm left}\,,\,W_{\rm right})$ *in the space-time plane.*

2) PARTIAL RIEMANN PROBLEM FOR HYPERBOLIC SYSTEMS.

• In this section we generalize in two directions the notion of Riemann problem presented in the previous section for the particular case of gas dynamics. First we consider a general nonlinear system of conservation laws and second we introduce the notion of **partial Riemann problem** between a state and a manifold.

2.1 **Simple waves for an hyperbolic system of conservation laws.**

• We study a system of conservation laws in one space dimension. The unknown function $\mathbb{R} \times [0,\, +\infty[\, \ni (x,\, t) \longmapsto W(x,\, t) \in \Omega$ takes its values inside a convex open cone Ω included in $\mathbb{R}^m$ (m is a fixed positive integer and $m = 3$ in the particular case of the Euler equations of gas dynamics studied previously) :

(2.1.1) $\quad \forall\, W \in \Omega\,, \quad \forall\, \lambda > 0\,, \quad \lambda\, W \in \Omega\,, \qquad \Omega \subset \mathbb{R}^m\,.$

The flux function $\Omega \ni W \longmapsto F(W) \in \mathbb{R}^m$ is supposed to be sufficiently regular (of $\mathcal{C}^2$ class typically). The conservation law takes the following classical form (see *e.g.* Godlewski-Raviart [GR96]) :

(2.1.2) $$\frac{\partial}{\partial t} W(x,\, t) \,+\, \frac{\partial}{\partial x}\, F(W(x,\, t)) \;=\; 0\,.$$

We suppose that the system of conservation laws (2.1.2) is a strictly hyperbolic system, that is for each $W \in \Omega$, there exists m real distinct eigenvalues $\lambda_j(W)$ satisfying conventionnaly the ordering condition

(2.1.3) $\lambda_1(W) \,<\, \lambda_2(W) \,<\, \cdots \,<\, \lambda_m(W)\,, \qquad W \in \Omega$

and associated with eigenvectors $R_j(W) \in \mathbb{R}^m$:

(2.1.4) $\mathrm{d}F(W) \bullet R_j(W) \,=\, \lambda_j(W)\, R_j(W)\,, \qquad W \in \Omega\,, \quad j \,=\, 1\,,\cdots,m\,.$

Each vector $r \in \mathbb{R}^m$ can be decomposed in the basis of eigenvectors $R_j(W)$ and the coordinates in this basis define the family of left-eigenvectors $\big(l_j(W)\big)_{j=1,\cdots m}$ which is exactly the dual basis of system $\big(R_j(W)\big)_{j=1,\cdots m}$:

(2.1.5) $$r \,\equiv\, \sum_{k=1}^{k=m} \big(l_j(W)\,,\, r\big)\, R_j(W)$$

with the classical property :

(2.1.6) $\big(l_j(W)\,,\, R_k(W)\big) \,=\, \delta_{jk}\,.$

Hypothesis 1. Genuinely nonlinear or linearly degenerate fields. We restrict ourselves to systems of conservation laws such that for each integer j with $1 \leq j \leq m$, the j-th field is supposed to be

- either genuinely nonlinear, that is

(2.1.7) $\mathrm{d}\lambda_j(W) \bullet R_j(W) \equiv 1\,, \quad \forall\, W \in \Omega \quad$ (j-th field genuinely nonlinear)

- either linearly degenerate, *id est*

(2.1.8) $\mathrm{d}\lambda_j(W) \bullet R_j(W) \equiv 0\,, \quad \forall\, W \in \Omega \quad$ (j-th field linearly degenerate).

• When the j-th field is genuinely nonlinear, it is possible to construct the so-called $j-$wave issued from a particular state $W_0 \in \Omega$. This (genuinely) nonlinear wave can be considered from two points of view. On the first hand, it is a curve $\epsilon \longmapsto \chi_j(\epsilon\,;\, W_0)$ inside the space Ω of all the states and on the other hand for real variable ϵ fixed sufficiently small, it is possible to construct a self-similar weak solution of the conservation law (2.1.2) between state W_0 and state $\chi_j(\epsilon\,;\, W_0)$.

• When $\epsilon > 0$ this particular wave is a $j-$**rarefaction** wave and is defined as an integral curve of the vector field $\Omega \ni W \longmapsto R_j(W) \in \mathbb{R}^m$ in the space of states :

(2.1.9) $$\frac{\partial}{\partial \epsilon}\big(\chi_j(\epsilon\,;\, W_0)\big) \,=\, R_j\big(\chi_j(\epsilon\,;\, W_0)\big)\,, \quad \epsilon > 0$$

(2.1.10) $\chi_j(0,\, W_0) \quad = \, W_0\,.$

In space-time plane, this $j-$curve allows to construct a continuous rarefaction, which is a particular solution of the conservation law (2.1.2) :

(2.1.11) $$W(x,\,t) \,=\, W_0 \qquad \text{if} \quad \frac{x}{t} \,<\, \lambda_j(W_0)$$

(2.1.12) $$W(x,\,t) \,=\, \chi_j\Big(\frac{x}{t} - \lambda_j(W_0)\,;\, W_0\Big) \quad \text{if } \lambda_j(W_0) \,\leq\, \frac{x}{t} \,\leq\, \lambda_j\big(\chi_j(\epsilon\,;\, W_0)\big)$$

(2.1.13) $$W(x,\,t) \,=\, \chi_j(\epsilon\,;\, W_0) \qquad \text{if} \quad \frac{x}{t} \,>\, \lambda_j\big(\chi_j(\epsilon\,;\, W_0)\big)\,.$$

Moreover we have

(2.1.14) $$\lambda_j\big(W(x,\,t)\big) \,\equiv\, \frac{x}{t} \qquad \text{if} \quad \lambda_j(W_0) \,\leq\, \frac{x}{t} \,\leq\, \lambda_j\big(\chi_j(\epsilon\,;\, W_0)\big)\,.$$

• When $\epsilon < 0$ the j-th nonlinear wave is a $j-$**shock** wave of celerity $\sigma_j(\epsilon\,;\,W_0)$ satisfying the entropy condition (*e.g.* in the sence of Lax [Lax73]). The states W_0 and $\chi_{_j}(\epsilon\,;\,W_0)$ are linked with celerity $\sigma_j(\epsilon\,;\,W_0)$ in state space according to the Rankine-Hugoniot jump conditions :

(2.1.15) $\quad F\big(\chi_{_j}(\epsilon\,;\,W_0)\big) \,-\, F(W_0) \,\equiv\, \sigma_j(\epsilon\,;\,W_0)\,\big(\chi_{_j}(\epsilon\,;\,W_0) \,-\, W_0\big)\,, \quad \epsilon < 0\,.$

In space-time space, this discontinuous self-similar $j-$shock wave $\mathbb{R} \times [0,\,+\infty[\; \ni (x,\,t) \longmapsto W(x,\,t) \in \Omega$ of strength $|\,\epsilon\,|$ is characterized by the following two conditions

(2.1.16) $\quad W(x,\,t) \,=\, W_0 \qquad\qquad \text{if } \dfrac{x}{t} < \sigma_j(\epsilon\,;\,W_0)$

(2.1.17) $\quad W(x,\,t) \,=\, \chi_{_j}(\epsilon\,;\,W_0) \qquad \text{if } \dfrac{x}{t} > \sigma_j(\epsilon\,;\,W_0)\,.$

• When the j-th field is linearly degenerate, the construction of the $j-$wave issued from the particular state $W_0 \in \Omega$ is still possible. Due to the condition (2.1.8), the $j^{\rm o}$ eigenvalue $\lambda_j(W)$ is constant along the integral curve defined in relations (2.1.9) (2.1.10) and we have also all along this curve the following jump relation :

(2.1.18) $\quad F\big(\chi_{_j}(\epsilon\,;\,W_0)\big) \,-\, F(W_0) \,\equiv\, \lambda_j(\epsilon\,;\,W_0)\,\big(\chi_{_j}(\epsilon\,;\,W_0) \,-\, W_0\big)\,,\ \forall\,\epsilon\,.$

In space-time plane, a self-similar $j-$**contact discontinuity** can be constructed between states W_0 and $\chi_{_j}(\epsilon\,;\,W_0)$ and we have :

(2.1.19) $\quad W(x,\,t) \,=\, W_0 \qquad\qquad \text{if } \dfrac{x}{t} < \lambda_j(\epsilon\,;\,W_0)$

(2.1.20) $\quad W(x,\,t) \,=\, \chi_{_j}(\epsilon\,;\,W_0) \qquad \text{if } \dfrac{x}{t} > \lambda_j(\epsilon\,;\,W_0)\,.$

All this material is summarized in the following result (see *e.g.* [GR96]).

Proposition 1.

We suppose that the hyperbolic system of conservation laws (2.1.2) satisfies Hypothesis 1. Then for each state $W_0 \in \Omega$ and for each integer j $(0 \leq j \leq m)$, there exists some vicinity Θ_j of 0 in $\mathbb{R}$ and there exists a $j-$wave $\chi_{_j} \,:\, \Theta_j \times \Omega \ni (\epsilon\,,$
$W_0) \longmapsto \chi_{_j}(\epsilon\,;\,W_0) \in \Omega$ which is a regular (of $\mathcal{C}^2$ class) curve. This corresponds to a $j-$rarefaction when the $j^{\rm o}$ field is genuinely nonlinear and $\epsilon > 0$, to an admissible discontinuity (weak shock satisfying an entropy condition) of conservation law (2.1.2) when the $j^{\rm o}$ field is genuinely nonlinear and $\epsilon < 0$ or to a $j-$contact discontinuity when the $j^{\rm o}$ field is linearly degenerate. Moreover, the $j^{\rm o}$ wave $\chi_{_j}(\bullet\,;\,W_0)$ is continuously derivable at starting point W_0 and we have :

(2.1.21) $\quad \chi_{_j}(\epsilon\,;\,W_0) \,=\, W_0 \,+\, \epsilon\, R_j(W_0) \,+\, O\big(\epsilon^2\big)\,, \quad \epsilon \in \Theta_j\,.$

• We remark that in space-time, a j-wave defines an entropy solution of hyperbolic system (2.1.1).

2.2 Classical Riemann problem between two states.

• We consider the solution of the particular Cauchy problem associated with the conservation law (2.1.1) and the particular initial datum

$$W(x\,,\,0) \;=\; \begin{cases} W_l\,, & x\,<\,0 \\ W_r\,, & x\,>\,0\,. \end{cases} \tag{2.2.1}$$

composed by two constant states on each side of a discontinuity located at $x = 0$. In what follows, this problem is refered as the **classical Riemann problem** and is denoted by $R(W_l\,,\,W_r)$. We have the following Theorem due to Lax (see *e.g.* [Lax 73]).

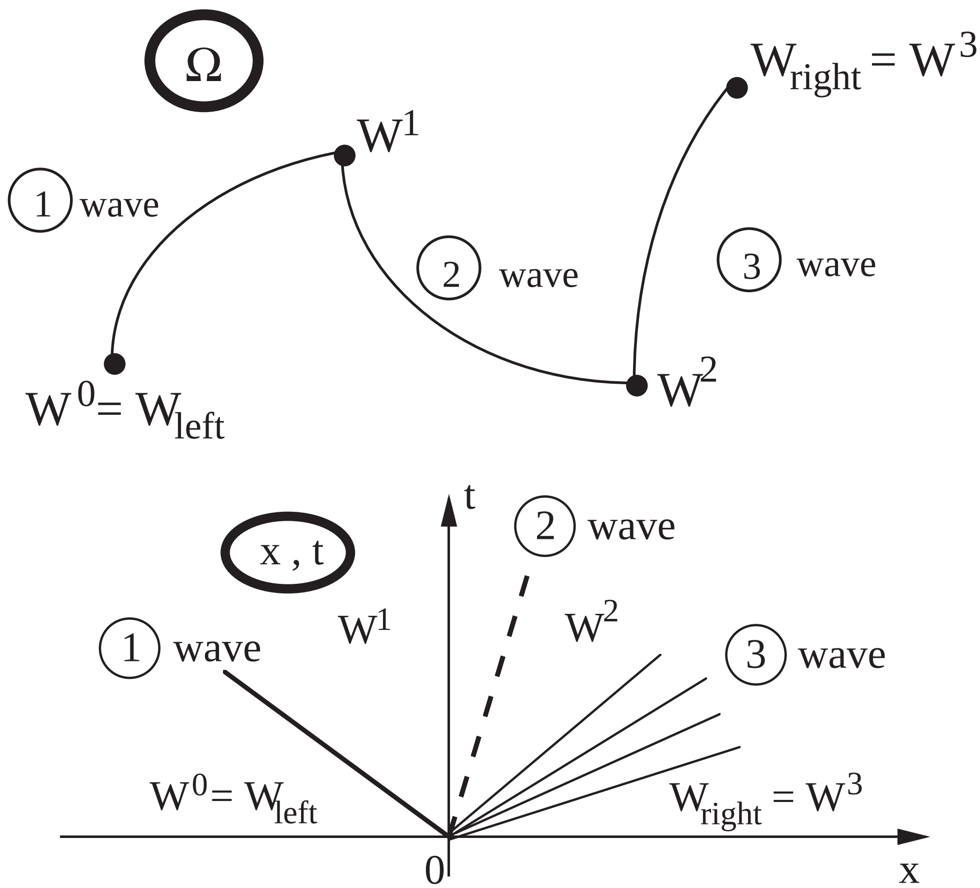

Figure 2.1 *Solution of the Riemann problem in the state space* Ω *and in the space-time plane.*

Theorem 1. Solution of the classical Riemann problem.
We suppose that the hyperbolic system of conservation laws (2.1.2) satisfies Hypothesis 1 : for $j \in \{1, \cdots, m\}$, the $j^{\rm o}$ field is either genuinely nonlinear or linearly degenerate. Then for each $W_l \in \Omega$ there exists a vicinity $\mathcal{Y}$ of state W_l ($\mathcal{Y} \in \mathcal{V}(W_l)$, set of all vicinities of the particular state W_l) such that for each state W_r lying in $\mathcal{Y}$ ($W_r \in \mathcal{Y}$), the Riemann problem $R(W_l\,, W_r)$ has a unique entropy solution composed by at most $(m+1)$ states separated by (at most) m elementary waves.

• Figure 2.1 shows what has to be done for the resolution of the classical Riemann problem in the particular case $m = 3$ that corresponds to the Euler equations of gas dynamics in one space dimension. The two intermediate states W^1 and W^2 are linked with data W_l and W_r acording to the wave relations :

(2.2.2) $$W^1 \;=\; \chi_{_1}(\epsilon_{_1}\,;\, W_l)$$

(2.2.3) $$W^2 \;=\; \chi_{_2}(\epsilon_{_2}\,;\, W^1)$$

(2.2.4) $$W_r \;=\; \chi_{_3}(\epsilon_{_3}\,;\, W^2)$$

and by elimination of the two intermediate states W^1 and W^2 we obtain

(2.2.5) $$\chi_{_3}\Big(\epsilon_{_3}\,;\, \chi_{_2}\big(\epsilon_{_2}\,;\, \chi_{_1}(\epsilon_{_1}\,;\, W_l)\big)\Big) \;=\; W_r$$

which is a set of three equations with three scalar unknowns $\epsilon_{_1}, \epsilon_{_2}$ and $\epsilon_{_3}$. When parameters $\epsilon_{_j}$ are determined (in practice for gas dynamics with the method presented in the previous section) each wave is acting in space-time space as presented on Figure 2.1.

• The proof of Theorem 1 in the general case consists in studying the chaining of m elementary waves, *i.e.* the mapping χ defined by the relations

(2.2.6) $$\left\{ \begin{array}{l} \mathbb{R}^m \;\supset\; \Theta_1 \times \cdots \times \Theta_m \;\ni\; (\epsilon_{_1}\,, \cdots, \epsilon_{_m}) \equiv \epsilon \longmapsto \chi(\epsilon) \in \Omega \\ \chi(\epsilon) \equiv \chi_m\big(\epsilon_{_m}\,;\, \chi_{_{m-1}}\big(\epsilon_{_{m-1}}\,;\, \cdots\,;\, \chi_1(\epsilon_{_1}\,;\, W_l)\cdots\big)\big)\,. \end{array} \right.$$

Two local properties have to be derived concerning on one hand the mapping χ itself at the origin :

(2.2.7) $$\chi(0) \;=\; W_l$$

and on the other hand the tangent vector field $\mathrm{d}\chi$ considered at the same point has a very simple expression :

(2.2.8) $$\mathrm{d}\chi(0) \bullet \eta \;=\; \sum_{j=1}^{m} \eta_j\, R_j(W_l)\,, \qquad \eta \;=\; \big(\eta_1, \cdots, \eta_m\big) \in \mathbb{R}^m\,.$$

Then the local inversion theorem proves that the equation

(2.2.9) $$\chi(\epsilon) \;=\; W_r$$

has a unique solution. For the details of this proof, we refer to Godlewski and Raviart [GR96]. □

• In space-time domain, the entropic selfsimilar solution $\mathbb{R} \times \,]0, +\infty[\ni (x, t) \mapsto U(\frac{x}{t} \,;\, W_l, W_r)$ of the Riemann problem $R(W_l, W_r)$ is constructed as follows. Let

$$\text{(2.2.10)} \qquad \begin{cases} W^0 = W_l \,, \cdots, \; W^j = \chi_j(\epsilon_j \,;\, W^{j-1}) \,, \cdots, \\ \qquad\qquad\qquad\qquad \cdots, W^m = \chi_m(\epsilon_m \,;\, W^{m-1}) = W_r \end{cases}$$

be the intermediate states, $\mu_j^-(W_l, W_r)$ the smallest wave celerity of the j^o wave and $\mu_j^+(W_l, W_r)$ the corresponding maximal wave celerity. If the j^o wave is a rarefaction, we have from (2.1.11) and (2.1.13) :

$$\text{(2.2.11)} \qquad \mu_j^-(W_l, W_r) = \lambda_j(W^{j-1}) \,, \quad \mu_j^+(W_l, W_r) = \lambda_j(W^j)$$

whereas in case of a j-shock wave or j-contact discontinuity, we have due to (2.1.16) and (2.1.17) :

$$\text{(2.2.12)} \qquad \mu_j^-(W_l, W_r) = \mu_j^+(W_l, W_r) = \sigma_j(\epsilon_j \,;\, W^{j-1}) \,.$$

Then the selfsimilar solution $U(\xi \,;\, W_l, W_r)$ satisfies

$$\text{(2.2.13)} \quad U(\xi \,;\, W_l, W_r) = \begin{cases} W^0 = W_l \,, & \xi < \mu_1^-(W_l, W_r) \\ \vdots & \\ \chi_j\Big(\dfrac{x}{t} - \mu_j^-(W_l, W_r)\Big), & \\ & \mu_j^-(W_l, W_r) < \xi < \mu_j^+(W_l, W_r) \\ W^j \,, & \mu_j^+(W_l, W_r) < \xi < \mu_{j+1}^-(W_l, W_r) \\ \vdots & \\ W^m = W_r \,, & \xi > \mu_m^+(W_l, W_r) \,. \end{cases}$$

2.3 Boundary manifold.

• The Riemann problem is a usefull tool to prescribe weakly a boundary condition for an hyperbolic system of conservation laws when a right state W_r is supposed to be given (see Section 3.3). For physically relevant conditions in gas dynamics, the data at the boundary are of the type "the pressure is known" or "total pressure and total temperature are given", and in consequence do not define explicitely a single state W_r. Nevertheless, a **manifold** of states is associated with these sets of incomplete **boundary** data, as we will develop in Section 3.4. This physical situation motivates the following definition of a boundary manifold.

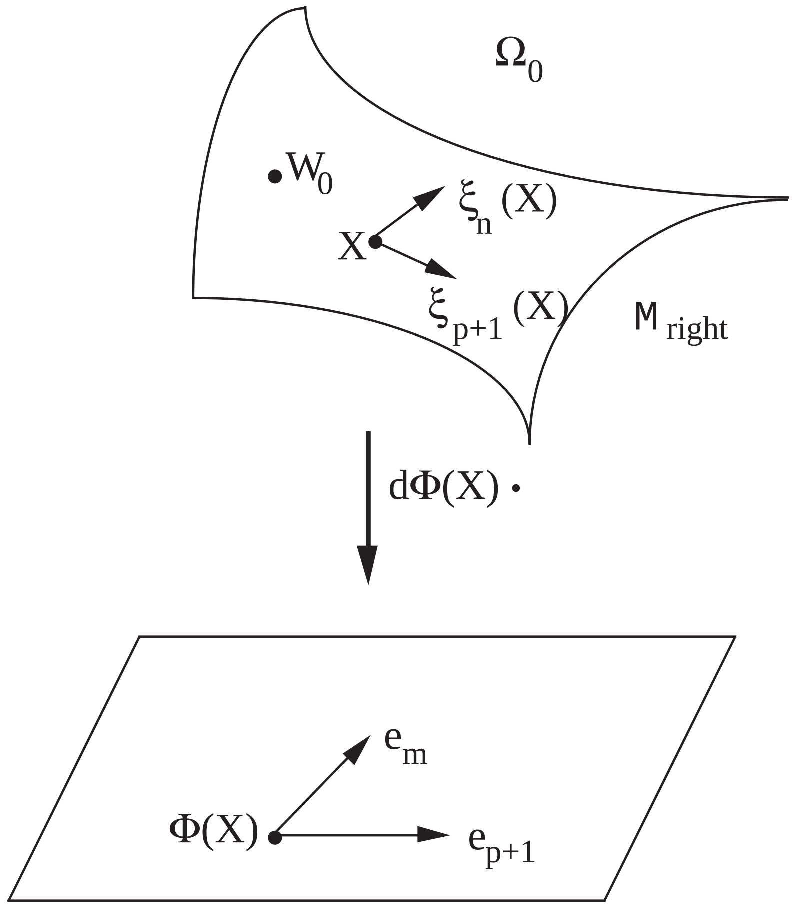

Figure 2.2 *Tangent vectors at the particular state X of the boundary manifold $\mathcal{M}_{\rm right}$.*

• We restrict ourselves to a local manifold $\mathcal{M}_r$ around a given state W_0. Consider a state $W_0 \in \Omega$, a vicinity Ω_0 of this state, *i.e.*

(2.3.1) $\quad W_0 \in \Omega \,, \quad \Omega_0 \in \mathcal{V}(W_0) \,, \quad \Omega_0 \subset \Omega \subset \mathbb{R}^m$

and an invertible regular local chart Φ defined on the vicinity Ω_0 and taking its values in some vicinity $\Theta \subset \mathbb{R}^m$ of 0 in $\mathbb{R}^m$:

(2.3.2) $\quad \Omega_0 \ni X \longmapsto \Phi(X) \in \Theta \subset \mathbb{R}^m \,, \quad \Theta \in \mathcal{V}(0) \,.$

Mapping Φ is one to one, of $\mathcal{C}^1$ class ($\Phi \in \mathcal{C}^1(\Omega_0,\, \Theta)$) and its inverse mapping Φ^{-1} is also of $\mathcal{C}^1$ class ($\Phi^{-1} \in \mathcal{C}^1(\Theta,\, \Omega_0)$). The boundary manifold $\mathcal{M}_{right} \equiv \mathcal{M}_r$ is here defined **locally** as the set of states X in the vicinity of Ω_0 ($X \in \Omega_0$) satisfying the equations

(2.3.3) $\quad \Phi_1(X) \,=\, \cdots \,=\, \Phi_p(X) \,=\, 0 \,, \quad X \in \Omega_0 \,.$

The index p is a fixed integer such that $0 \leq p \leq m$ and is the co-dimension of the boundary manifold $\mathcal{M}_r$. Recall that for each state X (or point $X \in \mathcal{M}_r$), the vector $\Phi(X)$ in $\mathbb{R}^m$ has its p first coordinates equal to zero :

(2.3.4) $\quad \Phi(\mathcal{M}_r) \,\subset\, \big\{ (0,\, \cdots,\, 0,\, y),\;\, y \in \mathbb{R}^{m-p} \big\} \,.$

• A system of tangent vector fields $\big(\xi_{p+1}(X)\,,\,\cdots\,,\,\xi_m(X)\big)$ at point $X \in \mathcal{M}_r$ is obtained by lifting the tangent mapping $\mathrm{d}\Phi(X)$ at point $X \in \Omega_0$. Let e_k be the k^{o} vector of the canonical basis in linear space $\mathbb{R}^m$. We have by definition :

(2.3.5) $\quad \mathrm{d}\Phi(X)\bullet\xi_k(X) \;=\; e_k \;,\quad X \in \mathcal{M}_r \;,\quad k \geq p+1$

as illustrated on Figure 2.2. We make a new hypothesis.

Hypothesis 2. Transversality.

We suppose that the following family $\Sigma(W_0)$ of m vectors :

(2.3.6) $\quad \Sigma(W_0) \;\equiv\; \big(R_1(W_0)\,,\,\cdots\,,\,R_p(W_0)\,,\,\xi_{p+1}(W_0)\,,\,\cdots\,,\,\xi_m(W_0)\big)$

is a basis of linear space $\mathbb{R}^m$.

2.4 Partial Riemann problem between a state and a manifold.

• The partial Riemann problem $P(W_l,\,\mathcal{M}_r)$ between a state W_l and a manifold $\mathcal{M}_r$ is by definition the Cauchy problem for the hyperbolic system of conservation laws (2.1.2) associated with the following constraints relative to the initial condition :

(2.4.1) $$W(x\,,\,0)\;\begin{cases} =\; W_l\;, & x\;<\;0\\ \in \mathcal{M}_r\;, & x\;>\;0\,.\end{cases}$$

The above definition of the partial Riemann problem $P(W_l,\,\mathcal{M}_r)$ has been first proposed in a particular case in [Du87]. We have also used it in [Du88], [DLF89], [DLL91] and [CDV92]. We first remark that if $p = m$ and $\mathcal{M}_r = \{W_r\}$, then the partial Riemann problem $P(W_l,\,\mathcal{M}_r)$ reduces to the classical Riemann problem $R(W_l,\,W_r)$. The following result has been first presented in [Du98].

Theorem 2. Existence of a solution for the partial Riemann problem.

We suppose that the hyperbolic system of conservation laws (2.1.2) satisfies Hypothesis 1 and that the boundary manifold $\mathcal{M}_r$ is defined as above with a given state $W_0 \in \Omega$ that satisfies Hypothesis 2, a vicinity Ω_0 of W_0 ($\Omega_0 \in \mathcal{V}(W_0)$) and a local chart $\Omega_0 \ni X \longmapsto \Phi(X) \in \Theta \subset \mathbb{R}^m$, $\Theta \in \mathcal{V}(0)$ and satisfying the relations (2.3.3)-(2.3.4). Then there exists a vicinity $\Omega_1 \subset \Omega_0$ of state W_0 such that for every state $W_l \in \Omega_1$, the partial Riemann problem $P(W_l,\,\mathcal{M}_r)$ defined by (2.1.2) and (2.4.1) has an entropy solution in space Ω composed by at most $(p+1)$ states

(2.4.2) $$\begin{cases} W_l \;=\; W^0\,,\; W^1 \;=\; \chi_{_1}(\epsilon_{_1}\,,\,W^0)\,,\;\cdots\,,\; W^{p-1} \;=\; \chi_{_{p-1}}(\epsilon_{_{p-1}}\,,\,W^{p-2}),\\ W^p \;=\; \chi_{_p}(\epsilon_{_p}\,,\,W^{p-1})\,, \qquad W^p \in \mathcal{M}_r,\end{cases}$$

separated by (at most) p simple waves. In space-time, the solution $\mathbb{R}\times]0,\,+\infty[\; \ni (x,\,t) \longmapsto U(\frac{x}{t}\,;\,W_l,\,\mathcal{M}_r)$ is obtained by superposition of the p simple waves considered previously. With the notations proposed in (2.2.11) and (2.2.12), we have

$$
U(\xi\,;\,W_l,\,W_r) \;=\; \begin{cases} W^0 \;=\; W_l\,, & \xi < \mu_1^- \\ \vdots & \\ \chi_j\Big(\dfrac{x}{t}-\mu_j^-\Big), & \mu_j^- < \xi < \mu_j^+ \\ W^j\,, & \mu_j^+ < \xi < \mu_{j+1}^- \\ \vdots & \\ W^p \,\in\, \mathcal{M}_r\;, & \xi > \mu_p^+\,. \end{cases} \tag{2.4.3}
$$

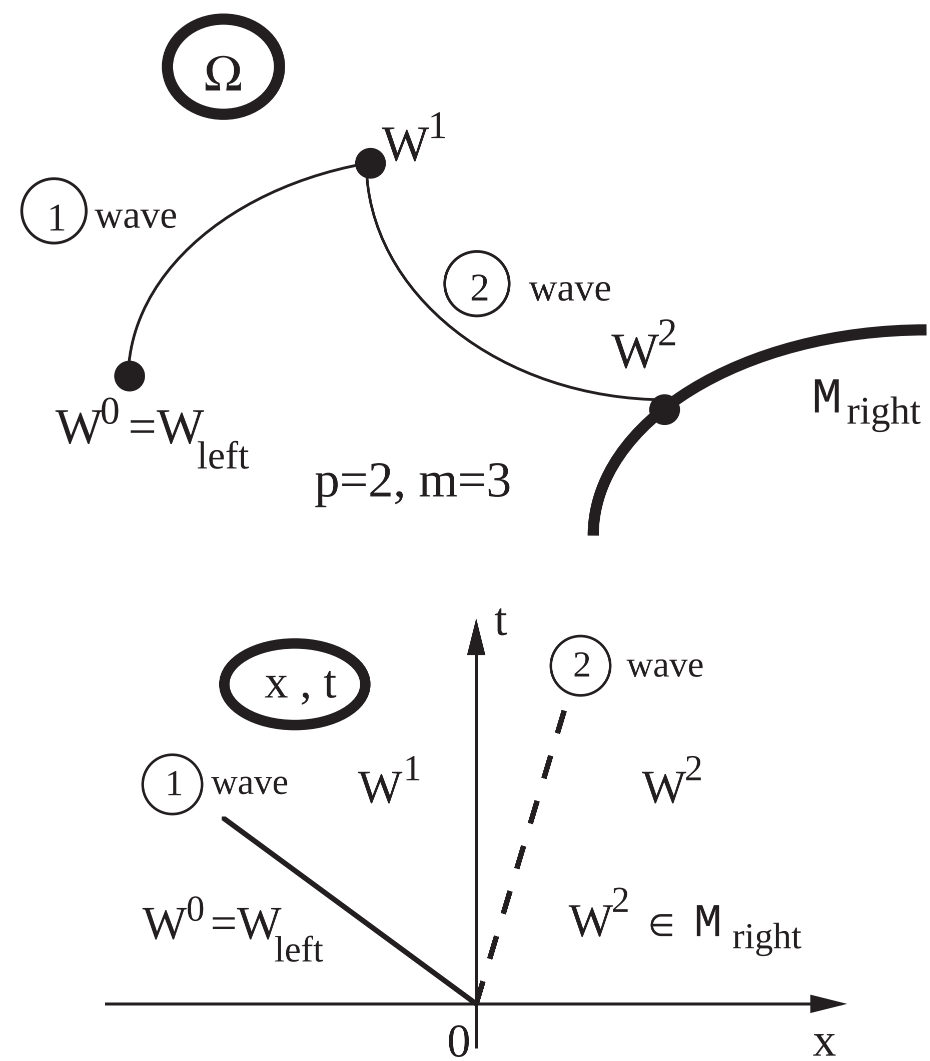

Figure 2.3 *Resolution of the partial Riemann problem* $P(W_{\rm left},\,\mathcal{M}_{\rm right})$ *in the state space* Ω *and in the space-time plane for a manifold* $\mathcal{M}_{\rm right}$ *of codimension* $p = 2$.

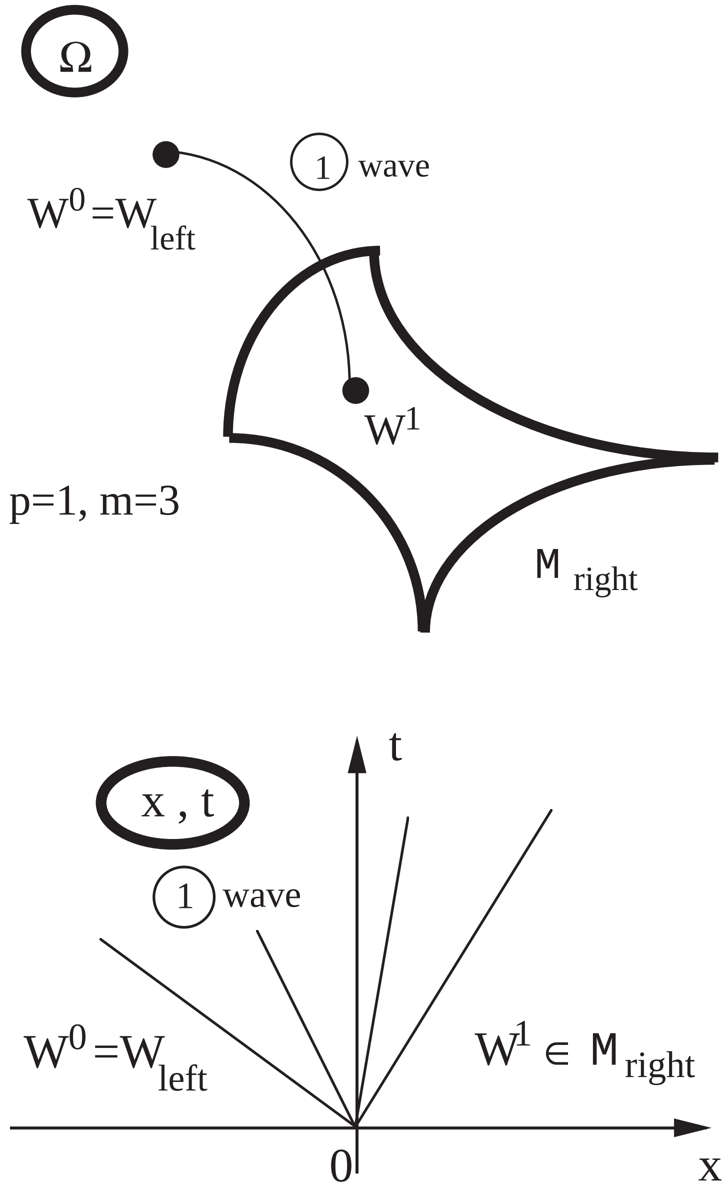

Figure 2.4 *Resolution of the partial Riemann problem* $P(W_{\rm left}, \mathcal{M}_{\rm right})$ *in the state space* Ω *and in the space-time plane for a manifold* $\mathcal{M}_{\rm right}$ *of codimension* $p = 1$.

• The proof of Theorem 2 follows the construction illustrated on Figures 2.3 and 2.4 for two particular cases. The idea is to apply the implicit function theorem to the mapping defined as follows. We first introduce a vicinity $\Delta = \Theta_1 \times \cdots \times \Theta_p$ of 0 in $\mathbb{R}^p$:

(2.4.4) $$\mathbb{R}^p \supset \Theta_1 \times \cdots \times \Theta_p \equiv \Delta \ni (\epsilon_1, \cdots, \epsilon_p) \equiv \epsilon, \qquad \Delta \in \mathcal{V}(0),$$

and a sub-vicinity $\Omega_1 \subset \Omega_0$ of given state W_0 in such a way that the mapping Ψ_p defined by the chaining of the **first** p **simple waves** is well defined :

(2.4.5) $$\left\{ \begin{array}{l} \Delta \times \Omega_1 \;\ni\; (\epsilon, W) \longmapsto \Psi_p(\epsilon, W) \;\in \Omega_0 \\ \Psi_p(\epsilon, W) \;=\; \chi_p \big(\epsilon_p \,; \chi_{p-1} \big(\epsilon_{p-1} \,; \cdots \,; \chi_1(\epsilon_1 \,; W) \cdots \big) \big) \,. \end{array} \right.$$

• We iterate this mapping with the p first components $\Phi_1, \cdots, \Phi_p$ of the local chart Φ defined at relations (2.3.2) to (2.3.4) :

(2.4.6) $$\left\{ \begin{array}{l} \Delta \times \Omega_1 \;\ni\; (\epsilon, W) \longmapsto \varphi(\epsilon, W) \;\in \mathbb{R}^p \\ \varphi(\epsilon, W) \;=\; \big(\Phi_1\big(\Psi_p(\epsilon, W)\big) , \cdots , \Phi_p\big(\Psi_p(\epsilon, W)\big) \big) \,. \end{array} \right.$$

We construct a solution of the partial Riemann problem composed by the conservation law (2.1.2) and the constraints (2.4.1) for the initial conditions with the p first waves issued from the left state W_l. In other terms, we search a right state $W_r \in \mathcal{M}_r$ as in the Riemann problem $R(W_l, W_r)$ *id est* under the form

(2.4.7) $$W_r \;=\; \Psi_p(\epsilon, W_l) \,, \qquad W_r \in \mathcal{M}_r \,.$$

The determination of the state W_r satisfying the conditions (2.4.7) is equivalent to the research of parameter $\epsilon \in \Delta \subset \mathbb{R}^p$ that satisfy the equation

(2.4.8) $$\varphi(\epsilon, W_l) \;=\; 0 \,.$$

• We have the following natural property :

(2.4.9) $$\Psi_p(0, W_0) \;=\; W_0 \in \mathcal{M}_r \,, \qquad \varphi(0, W_0) \;=\; 0 \;\in \mathbb{R}^p$$

and moreover from Hypothesis 2, the $m \times m$ matrix composed by the family $\Sigma(W_0) \;=\; \big(R_1(W_0) , \cdots , R_p(W_0) , \xi_{p+1}(W_0) , \cdots , \xi_m(W_0)\big)$ has a rank equal to m. Then the same property holds after applying the linear one to one mapping $\mathrm{d}\Phi(W_0)$:

(2.4.10) $$\mathrm{rank} \big(\mathrm{d}\Phi(W_0) \bullet R_1(W_0) , \cdots , \mathrm{d}\Phi(W_0) \bullet R_p(W_0) , e_{p+1} , \cdots , e_m \big) \;=\; m$$

due to the definition (2.3.5) of tangent vectors $\xi_{p+1}(W_0) , \cdots , \xi_m(W_0)$. In consequence, when we look to the matrix defined at the relation (2.4.10), we observe that the block composed by the p first lines and the p first columns at the top and the left of this matrix has a rank exactly equal to p. Moreover, this $p \times p$ matrix is exactly equal to $\frac{\partial \varphi}{\partial \epsilon}(0, W_0)$ and in consequence this jacobian matrix is invertible :

(2.4.11) $$\frac{\partial \varphi}{\partial \epsilon}(0, W_0) \;\text{ is an invertible }\; p \times p \;\text{ matrix.}$$

• Now the theorem of implicit functions proves, with the eventual constraint that vicinity Ω_1 may have to be reduced, that the equation

(2.4.12) $$\varphi(\epsilon, W) \;=\; 0$$

admits a unique solution (ϵ, W) in the vicinity of $(0, W_0)$ and it takes the form $\epsilon \;=\; \pi(W)$:

(2.4.13) $$\left\{ \begin{array}{l} \exists \;\Omega_1 \ni W \;\longmapsto\; \epsilon = \pi(W) \;\in \Delta \subset \mathbb{R}^p \,, \quad \pi \in \mathcal{C}(\Omega_1, \Delta) \;\text{ such that} \\ \big(\varphi(\epsilon, W) \;=\; 0 \,,\; W \in \Omega_1 \big) \;\Longrightarrow\; \big(\epsilon \;=\; \pi(W) \big) \,. \end{array} \right.$$

The general structure of the solution of equation (2.4.8) (or of the equivalent equation (2.4.12)) is a consequence of the implicit function theorem presented in

(2.4.13) : it allows the determination of the strengh ϵ of the waves as a function of left state W_l. Then Theorem 2 is established. $\square$

• The extension of the previous notion to a partial Riemann problem $P(\mathcal{M}_l,\ W_r)$ composed by a boundary manifold $\mathcal{M}_{\rm left} \equiv \mathcal{M}_l$ and a state $W_{\rm right} \equiv W_r$ is defined by the conservation law (2.1.2) and the initial conditions

$$W(x\,,\,0) \left\{ \begin{array}{ll} \in \mathcal{M}_l\ , & x\,<\,0 \\ =\ W_r\ , & x\,>\,0\,. \end{array} \right. \tag{2.4.3}$$

If $\mathrm{codim}(\mathcal{M}_l) = p$, the construction proposed at Theorem 2 can be extended without difficulty. For each state W_r sufficiently close to the manifold $\mathcal{M}_l$, the partial Riemann problem $P(\mathcal{M}_l,\, W_r)$ admits an entropy solution composed by the **last** p waves of the Riemann problem :

$$\left\{ \begin{array}{l} W^0 \in \mathcal{M}_l\,,\ W^1\,=\,\chi_{m-p+1}(\epsilon_{m-p+1}\,,\,W^0)\,,\ \cdots\,, \\ W^{p-1}\,\,=\,\,\chi_{m-1}(\epsilon_{m-1}\,,\,W^{p-2})\,,\ \ W^p\,\,=\,\,\chi_m(\epsilon_m\,,\,W^{p-1})\,=\,W_r\,. \end{array} \right. \tag{2.4.15}$$

2.5 Partial Riemann problem with an half-space.

• The notion of partial Riemann problem can be extended to the particular situation of a "half space" defined as follows. We first introduce the set $\mathcal{M}_{right} \equiv \mathcal{M}_r$ defined by

$$\mathcal{M}_r\;=\;\{\,W \in \Omega\,,\ \lambda_1(W)\,\geq\,0\,\}\,. \tag{2.5.1}$$

The partial Riemann problem $P(W_l\,,\,\mathcal{M}_r)$ between the particular state W_l and the half space $\mathcal{M}_r$ is still defined by the partial differential equation (2.1.2) and the initial constraints (2.4.1). We have the

Proposition 2.

Partial Riemann problem with a particular half space.

We suppose that the 1-field is genuinely nonlinear. A particular solution of partial Riemann problem (2.2.1)-(2.4.1) between left state W_l and right half space $\mathcal{M}_r$ defined in (2.5.1) can be constructed as follows :

• if $\lambda_1(W_l)\,\geq\,0$, then $W_l \in \mathcal{M}_r$ and $\mathbb{R}\times[0,\,+\infty[\,\ni(x,\,t)\longmapsto W(x,\,t)\in\Omega$ is equal to a constant state :

$$W(x,\,t)\;\equiv\;W_l\;, \qquad \lambda_1(W_l)\,\geq\,0\,. \tag{2.5.2}$$

• if $\lambda_1(W_l)\,<\,0$, the following rarefaction wave :

$$W(x,\,t)\;=\;\left\{ \begin{array}{ll} W_l & \text{if}\ \ \dfrac{x}{t}\,<\,\lambda_1(W_l) \\ \chi_1\Big(\dfrac{x}{t} - \lambda_1(W_l)\,,\,W_l\Big) & \text{if}\ \lambda_1(W_l)\,\leq\,\dfrac{x}{t}\,\leq\,0 \\ \chi_1\big(-\lambda_1(W_l)\,,\,W_l\big) & \text{if}\ \ \dfrac{x}{t}\,>\,0 \end{array} \right. \tag{2.5.3}$$

is an entropy solution of the partial Riemann problem (2.1.2) (2.4.1) (2.5.1).

• The proof of Proposition 2 is a direct consequence of the notion of rarefaction wave explicited at relations (2.1.11) to (2.1.13). When $\mathcal{M}_r$ is the half space of "supersonic outflow" ($\lambda_1 \equiv u - c \geq 0$ for the Euler equations of gas dynamics), the solution of the partial Riemann problem is trivial when state W_l satisfies this condition. On the contrary, the solution of the partial Riemann problem is constructed with the help of a 1-wave for linking the "subsonic left state" W_l ($u_l < c_l$) to the half space $\mathcal{M}_r$. The "end-point" of the rarefaction is the **sonic** state $W_r = \chi_{_1}\big(-\lambda_1(W_l)\,,\, W_l\big) \in \mathcal{M}_r$ for particular celerity $\frac{x}{t} = 0$ (see Figure 2.5). $\square$

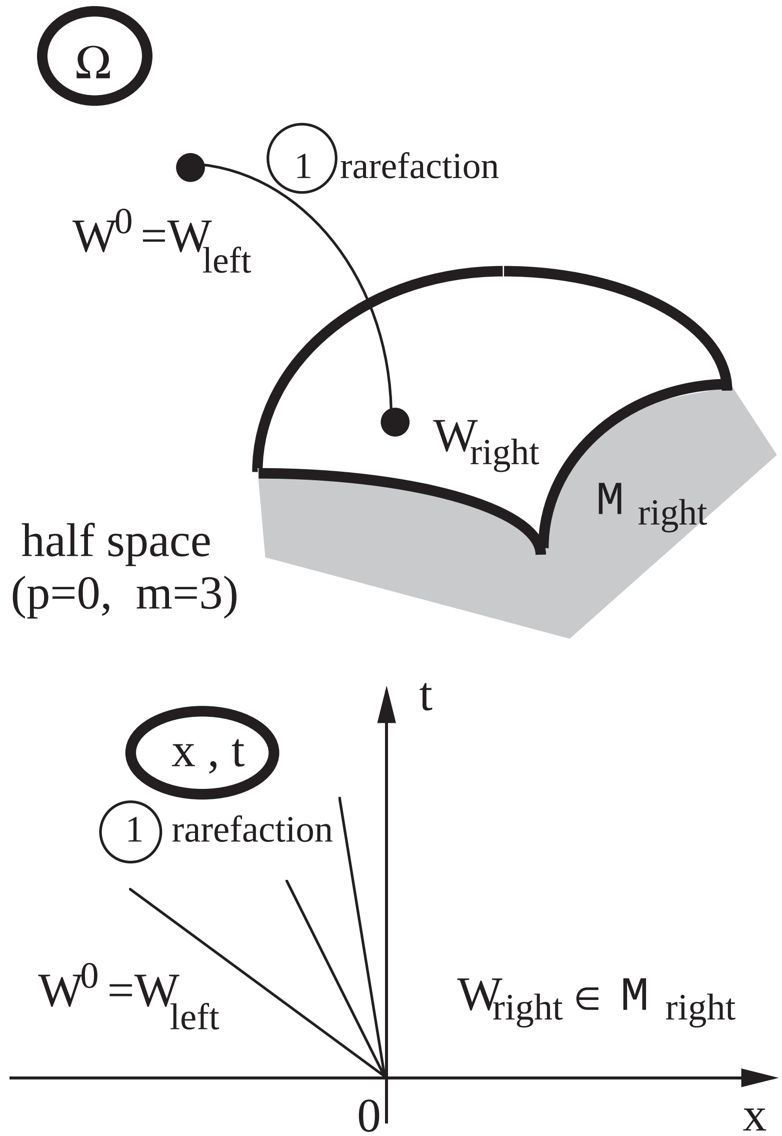

Figure 2.5 *Resolution of the partial Riemann problem* $P(W_{\rm left}, \mathcal{M}_{\rm right})$ *in the state space* Ω *and in the space-time plane for an half space* $\mathcal{M}_{\rm right}$ *describing a supersonic outflow.*

3) NONLINEAR BOUNDARY CONDITIONS FOR GAS DYNAMICS.

• In this section we use the notion of partial Riemann problem to consider boundary conditions for gas dynamics. We first make the link between this notion and the way that linear hyperbolic systems are well posed in the sense of least squares. Then we detail some cases where physically relevant boundary conditions can be interpreted with particular partial Riemann problems.

3.1 System of linearized Euler equations.

• Most of the known mathematical results concern hyperbolic **linear** equations. We linearize the Euler equations of gas dynamics around a **constant state** W_0 with density ρ_0, velocity u_0, pressure p_0 and sound celerity c_0. We set :

$$W \;=\; W_0 \,+\, W' \tag{3.1.1}$$

and we neglect second order terms relatively to the variable W'. In particular, the incremental variables $\rho' \equiv \rho - \rho_0$, $u' \equiv u - u_0$, $s' \equiv s - s_0$ define the incremental vector Z' of nonconservative variables :

$$Z' \;=\; \big(\rho' \,,\, u' \,,\, s'\big)^{\mathrm{t}} \tag{3.1.2}$$

and the difference of pressure $p' \equiv p - p_0$ is given at the first order as a function of the incremental thermodynamic variables ρ' and s' :

$$p' \;\equiv\; p - p_0 \;=\; c_0^2\,\rho' \,+\, \frac{\partial p}{\partial s}(W_0)\, s' \,. \tag{3.1.3}$$

Starting from expression (1.2.15) of the Euler equations of gas dynamics, we get at the same level of approximation :

$$\frac{\partial Z'}{\partial t} \,+\, B(W_0)\,\frac{\partial Z'}{\partial x} \;=\; 0\,. \tag{3.1.4}$$

• As in section 1.2, we diagonalize matrix $B(W_0)$ whose expression has been given at relation (1.2.14) and eigenvectors $\widetilde{R}_j(W_0)$ in (1.2.21). We can express the components φ_j of incremental vector Z' in the basis of vectors $\widetilde{R}_j(W_0)$. These variables φ_j are called the **characteristic variables** :

$$Z' \;=\; \sum_{j=1}^{3} \varphi_j \; \widetilde{R}_j(W_0) \;\equiv\; \varphi \bullet \widetilde{R}(W_0) \tag{3.1.5}$$

and we have from (1.2.21) the following expressions :

$$\left\{ \begin{aligned} \varphi_1 \;&=\; \frac{1}{2\,\rho_0\,c_0^2}\,\big(p' \,-\, \rho_0\,c_0\,u'\big) \\ \varphi_2 \;&=\; -\frac{1}{c_0^2}\,s' \\ \varphi_3 \;&=\; \frac{1}{2\,\rho_0\,c_0^2}\,\big(p' \,+\, \rho_0\,c_0\,u'\big)\,. \end{aligned} \right. \tag{3.1.6}$$

• The change of variables $\mathbb{R}^3 \ni Z' \longmapsto \varphi \in \mathbb{R}^3$ allows to decouple the system of equations (3.1.4) into three **uncoupled** advection equations :

$$\frac{\partial \varphi}{\partial t} + \Lambda(W_0)\,\frac{\partial \varphi}{\partial x} \,=\, 0 \qquad (3.1.7)$$

where $\Lambda(W_0) \equiv \mathrm{diag}\,(u_0 - c_0\,,\, u_0\,,\, u_0 + c_0)$. The system (3.1.7) is called the characteristic form of the linearized Euler equations. The above study motivates the mathematical study of linearized hyperbolic linear systems.

3.2 Boundary problem for linear hyperbolic systems.

• In a classical article, Kreiss [Kr70] developed the notion of **well posed problem** for initial boundary value problems in the quarter of space $x \leq L$ and $t \geq 0$ associated with linear hyperbolic systems of the type (3.1.4) or (3.1.7). We suppose that the boundary $x = L$ is **non-characteristic**, *id est*

$$u_0 - c_0 \,\neq\, 0\,, \quad u_0 \,\neq\, 0\,, \quad u_0 + c_0 \,\neq\, 0\,, \qquad (3.2.1)$$

and we denote by Λ_0^- (respectively Λ_0^+) the negative part (respectively the positive part) of the nonsingular matrix $\Lambda(W_0)$. We decompose also the characteristic vector φ into the ingoing components φ^- and the outgoing components φ^+ :

$$\begin{cases} \varphi \,\equiv\, \varphi^- \,+\, \varphi^+ \\ \varphi^- \,=\, \big\{\, (\varphi_j^-)_{1 \leq j \leq 3}\,,\ \varphi_j^- \,=\, \varphi_j \ \text{ if } \lambda_j(W_0) < 0,\, \varphi_j^- \,=\, 0 \text{ if } \lambda_j(W_0) > 0 \,\big\} \\ \varphi^+ \,=\, \big\{\, (\varphi_j^+)_{1 \leq j \leq 3}\,,\ \varphi_j^+ \,=\, \varphi_j \ \text{ if } \lambda_j(W_0) > 0,\, \varphi_j^+ = 0 \text{ if } \lambda_j(W_0) < 0 \,\big\}\,. \end{cases} \qquad (3.2.2)$$

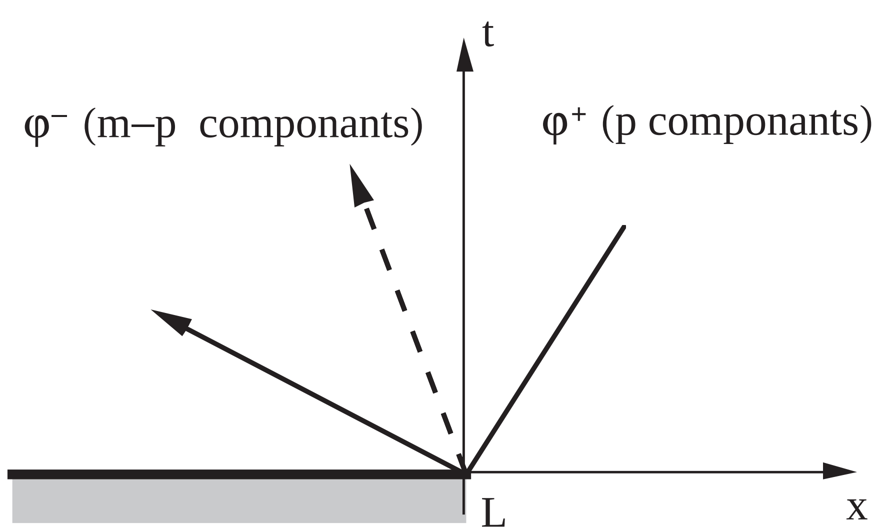

Figure 3.1 *Characteristic directions at the boundary* $x = L$ *of the domain* $[0,\, L]$.

Due to our choice to consider the exterior of the domain “at the right” of the domain $\{x \leq L\}$ of study, the vector φ^- is associated with negative eigenvalues of matrix $\Lambda(W_0)$ and vector φ^+ corresponds to positive eigenvalues of the same matrix. With the above notations, Kreiss has proved [Kr70] that in very general situations, the linear hyperbolic system associated with the initial condition

(3.2.3) $\varphi(x,\,0) \;=\; \varphi_0(x)\,, \qquad x \leq L$

and the boundary condition parameterized by the reflection operator Σ that associates to the outgoing characteristics variables φ^+ a linear function $\Sigma \bullet \varphi^+$ which is an incoming caracteristic variable :

(3.2.4) $\varphi^- \;=\; \Sigma \bullet \varphi^+ \;+\; g\,, \qquad x = L$

is **well posed** in space L^2.

• The boundary condition (3.2.4) can be interpreted in terms of **characteristic directions** : the field φ^- along the boundary is an **affine function** of the outgoing characteristics φ^+ (see Figure 3.1). The previous result remains true in more complicated situations. In particular, the multidimensional case can be considered with the same arguments except that direction x has to be replaced by the normal direction (see *e.g.* Higdon [Hi86]). In the characteristic case where *e.g.* the reference velocity u_0 is null, Majda and Osher [MO75] have extended Kreiss' result. We remark also that the boundary condition can naturally be written in terms of a **boundary manifold**, even if it is an affine manifold in the present case. We introduce the incremental input vector Z^- and the associated output vector Z^+ :

(3.2.5)
$$\left\{ \begin{array}{lcl} Z' & \equiv & Z^- + Z^+ \\ Z^- & = & \displaystyle\sum_{\lambda_j(W_0)<0} \varphi_j \, \widetilde{R}_j(W_0) \;=\; \sum_j \varphi_j^- \, \widetilde{R}_j(W_0) \\ Z^+ & = & \displaystyle\sum_{\lambda_j(W_0)>0} \varphi_j \, \widetilde{R}_j(W_0) \;=\; \sum_j \varphi_j^+ \, \widetilde{R}_j(W_0)\,, \end{array} \right.$$

and define the manifold $\mathcal{M}_{right} \equiv \mathcal{M}_r$ by the condition

(3.2.6) $\mathcal{M}_r \;=\; \big\{ \, Z' \text{ given by relations (3.2.5)}, \quad \varphi^- \;=\; \Sigma \bullet \varphi^+ \;+\; g \, \big\}$

and re-write the boundary condition (3.2.4) under the equivalent form :

(3.2.7) $Z' \,\in\, \mathcal{M}_r\,.$

• When we consider the boundary condition (3.2.7), we distinguish classically between four cases, following the sign of velocity u_0 and the modulus $|\,u_0\,|$ compared with sound celerity c_0. If $u_0 < 0$ at $x = L$, the fluid enters inside the domain $]-\infty,\,L]$ and if $u_0 > 0$ the boundary $\{x = L\}$ is an output. If $|\,u_0\,| < c_0$, the flow is subsonic whereas if $|\,u_0\,| > c_0$, it is supersonic.

(i) **Supersonic inflow** ($u_0 \,<\, -c_0$). The outgoing component of φ, *i.e.* φ^+ is reduced to zero, the manifold $\mathcal{M}_r$ is reduced to the unique point $g \bullet \widetilde{R}(W_0)$ and $\mathcal{M}_r$ is of codimension $p = 3$. The boundary condition (3.2.7) is equivalent to prescribe all the components of vector Z'.

(ii) **Subsonic inflow** ($-c_0 \,<\, u_0 \,<\, 0$). The number of incoming characteristics is two and there is one outgoing characteristic direction. The manifold $\mathcal{M}_r$ is of co-dimension $p = 2$. The linearized problem is well posed when any of the following pairs of **two** variables are given (see Oliger-Sundstrom [OS78]

or Yee-Beam-Warming [YBW82]) : (density, pressure), (velocity, pressure) or (enthalpy, entropy) ; we extend this context in the next section.

(iii) **Subsonic outflow** $(0 < u_0 < c_0)$. Only one characteristic is going inside the domain of study and two are going outside. The manifold $\mathcal{M}_r$ is of co-dimension 1 and only **one** relation has to be given between the scalar data on the boundary ; we remark that the classical choice of imposing the pressure $p = p_0$ can be written after linearization :

(3.2.8) $\qquad p' \,=\, 0\,.$

This condition is equivalent to choose

(3.2.9) $\qquad \Sigma \,=\, \big(\,0\,,\,-1\,\big)\,, \qquad g \,=\, 0$

inside relations (3.2.5) and (3.2.6) because the boundary condition (3.2.8) can also be written under the equivalent form :

(3.2.10) $\qquad \varphi_1 \,=\, -\varphi_3\,.$

(iv) **Supersonic outflow** $(u_0 > c_0)$. All the characteristic directions are going in the direction opposite to the domain $\{x \leq L\}$ (*id est* $\varphi^- = 0$) and condition (3.2.7) does not carry any information : **no numerical datum** has to be prescribed for a supersonic outflow.

• As a complement of the previous cases, we can add the important case where $u_0 = 0$ that corresponds physically to a **rigid boundary**. The linearized rigid wall boundary condition takes the form :

(3.2.11) $\qquad u' \,=\, 0$

and the latter boundary condition is equivalent to prescribe a **null** mass flux through the boundary for the linearized equations. Note that the previous condition (3.2.11) can also be written on the form (3.2.4) with the particular choice :

(3.2.12) $\qquad \Sigma \,=\, \big(\,0\,,\,1\,\big)\,, \qquad g \,=\, 0$

and conditions (3.2.11) (3.2.12) correspond to the relation

(3.2.13) $\qquad \varphi_1 \,=\, \varphi_3$

between the characteristic variables. It can be shown (*e.g.* Oliger-Sundstrom [OS78]) that the the initial boundary value problem (3.1.4), (3.2.3) and (3.2.12) is well posed with the natural L^2 condition.

3.3 Weak Dirichlet nonlinear boundary condition.

• For two particular systems of conservation laws of the type

(3.3.1) $$\frac{\partial W}{\partial t} + \frac{\partial}{\partial x}\, F(W) \,=\, 0\,,$$

i.e. for linear hyperbolic systems and nonlinear scalar conservation laws, following ideas developed by T. P. Liu ([Li77], [Li82]) for initial boundary value problems and transonic flow in nozzles, we have remarked with Le Floch in [DLF87] and [DLF88] that an efficient way to set a boundary condition of the

type "given state W_{right} outside the domain of study", say $\{x \leq L\}$ to fix the ideas, is to consider a **weak form** of the Dirichlet boundary condition $W = W_r$ in the following sense :

(3.3.2) $\qquad W(L^-, t) \in \mathcal{B}(W_r)\,.$

In relation (3.3.2), the state $W(L^-, t)$ is the limit value of internal state $W(x,\, t)$ as $x < L$ tends to the boundary, *i.e.* $x = L^-$ and $\mathcal{B}(W_r)$ is a set of **admissible states at the boundary** associated with datum W_0. Even if other formulations are possible and are still in development (see among others Nishida-Smoller [NS77], Bardos, Leroux and Nédélec [BLN79], Audounet [Au84], Benabdallah [Be86], Benabdallah-Serre [BS87], [DLF88], Bourdel, Delorme and Mazet [BDM89], Gisclon [Gi94], Serre [Se96]), we focus here on the use of the **Riemann problem** for taking into account a physical boundary conditon, in particuler for gas dynamics. We define the set of admissible states at the boundary as follows :

$$\mathcal{B}\big(W_r\big) = \left\{ \begin{array}{l} \text{values at } \ \frac{x}{t} = 0^- \ \text{ of the entropic solution} \\ \text{of the Riemann problem } \ R(W,\, W_r),\, W \in \Omega \end{array} \right\}. \tag{3.3.3}$$

• The advantage of the previous definition is that the initial boundary value problem parameterized by **constant** boundary datum W_r, **constant** initial condition W_0 and set on space-time domain $(x,\, t) \in]-\infty,\, L[\times[0,\, +\infty[$ by the conditions

$$\left\{ \begin{array}{llll} \dfrac{\partial W}{\partial t} + \dfrac{\partial}{\partial x} F(W) & = \ 0 & x < L\,, & t > 0 \\ W(x,\, 0) & = \ W_0 & x < L\,, & t = 0 \\ W(L^-,\, t) & \in \ \mathcal{B}\big(W_r\big) & t > 0 & \end{array} \right. \tag{3.3.4}$$

is **well posed** in conditions analogous to Theorem 1 for the Riemann problem in Lax's theory (see Section 2.2). Moreover, the set $\mathcal{B}\big(W_r\big)$ is easy to evaluate explicitely. In [DLF88], we have calculated the boundary set $\mathcal{B}\big(W_r\big)$ in the particular case of Euler-Saint Venant equations of isentropic gas dynamics. Even when the state W_r corresponds to a supersonic inflow in the linearized analysis, the admissible set $\mathcal{B}\big(W_r\big)$ is reduced to $\{W_r\}$ only in a vicinity of state W_r. For large differences between limit state $W(L^-,\, t)$ and weakly imposed boundary state W_r these two states can be linked together with an entire family of waves that compose the Riemann problem, and state $W(L^-,\, t)$ can even correspond to a supersonic outflow !

• We propose here to extend the previous **weak** boundary Dirichlet condition (3.3.4) to a **manifold** $\mathcal{M}_r$. We first introduce a new set $\beta\big(\mathcal{M}_r\big)$ of admissible states at the boundary :

$$\beta\big(\mathcal{M}_r\big) = \left\{ \begin{array}{l} \text{values at } \ \frac{x}{t} = 0^- \ \text{ of an entropic solution of the} \\ \text{partial Riemann problem } \ P(W,\, \mathcal{M}_r),\, W \in \Omega \end{array} \right\}. \tag{3.3.5}$$

We remark that this set of admissible states is a natural extension of definition

(3.3.3) relative to admissible states associated with a single state W_l. The initial boundary value problem associated with datum $\mathcal{M}_r$ is now formulated as follows :

$$
(3.3.6) \qquad \left\{ \begin{array}{lll} \dfrac{\partial W}{\partial t} + \dfrac{\partial}{\partial x} F(W) & = \; 0 & x < L \,, \qquad t > 0 \\ W(x,\, 0) & = \; W_0 & x < L \,, \qquad t = 0 \\ W(L^-,\, t) & \in \; \beta\big(\mathcal{M}_r\big) & t > 0 \,. \end{array} \right.
$$

• Theorem 2 shows that when constant initial datum W_0 and boundary manifold $\mathcal{M}_r$ are closed enough, the problem (3.3.6) is well posed in the family of solutions composed by the p "first" waves as described in relation (2.4.2). The solution of the initial boundary value problem (3.3.6) is the self-similar solution $(x,\, t) \longmapsto U(\frac{x-L}{t}\,;\, W_l,\, \mathcal{M}_r)$ described at relations (2.4.3) in Theorem 2. We remark that only the waves with negative celerities contribute to problem (3.3.6) even if the partial Riemann problem contains waves with positive celerities (see *e.g.* figure 2.3 or 2.4).

3.4 Some fluid boundary conditions.

• We show in this section that for a family of classical examples, it is possible to introduce a partial Riemann problem in order to consider fluid boundary conditions for the gas dynamics equations. We examine five particular cases : given state at infinity or supersonic inflow, subsonic inflow associated with a jet or a nozzle, subsonic pressure outflow of given static pressure and supersonic outflow. Even if we still adopt a classical denomination for these boundary conditions, the fact that the fluid enters into the domain ("inflow") or goes outside it ("outflow") has no influence for the classification of the boundary conditions. The construction of a link between physical data and mathematical model is done *via* a precise choice of a boundary manifold.

• **Given state.** This case corresponds typically to external aerodynamics problems. For this kind of flow, a state $W_r = W_\infty$ is known at a sufficient large distance between the object of study and the fluid boundary. In our present model this particular boundary is located at $x = L$. It is natural to set in a weak sense the boundary condition thanks to a (classical) Riemann problem and we have in this particular case a manifold $\mathcal{M}^{\rm state}_{W_\infty}$ reduced to a single state :

$$
(3.4.1) \qquad \mathcal{M}^{\rm state}_{W_\infty} \; \equiv \; \{\, W_\infty \,\} \,.
$$

We remark that this particular case is mathematically equivalent to the classical so-called "supersonic inflow" boundary condition. The knowledge of the entire state W_∞ is given and it interacts through the boundary with all the waves of the Riemann problem.

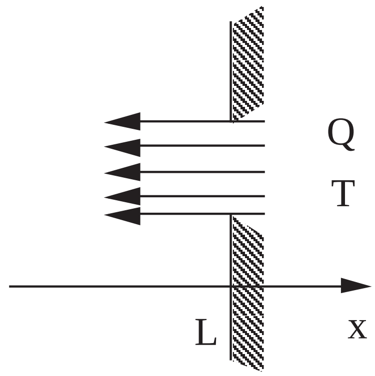

Figure 3.2 *Subsonic jet inflow at the boundary of the domain* $\{x \leq L\}$. *It is described by a manifold* $\mathcal{M}^{\rm jet}_{Q,T}$ *parameterized by mass flux* $Q < 0$ *and temperature* T.

• **Subsonic jet inflow.** In this case introduced in [CDV92] and described on Figure 3.2, we suppose that the fluid has a given flow rate Q and a given temperature T. We note that in order to respect the positive sign of the external normal of the boundary, the mass flux must be chosen negative $(Q < 0)$. We introduce the boundary manifold $\mathcal{M}^{\rm jet}_{Q,T}$ equal to the set of states W that respect exactly the boundary condition :

$$(3.4.2) \qquad \mathcal{M}^{\rm jet}_{Q,T} \equiv \Big\{ W = (\rho\,,\, q\,,\, \epsilon)^{\rm t} \in \Omega, \quad q \,=\, Q\,, \quad \epsilon \,=\, \rho\, C_v\, T \,+\, \frac{1}{2} \frac{Q^2}{\rho} \Big\}.$$

The resolution of the partial Riemann problem $P(W_l,\, \mathcal{M}^{\rm jet}_{Q,T})$ can be conducted in the plane of velocity and pressure as it is the case for the classical Riemann problem. We first express the internal energy e in terms of temperature T :

$$(3.4.3) \qquad e \,=\, C_v\, T$$

and extract density ρ from mass flow Q and velocity u :

$$(3.4.4) \qquad \rho \,=\, \frac{Q}{u}\,.$$

Then the thermostatic law (1.1.2) for polytropic perfect gas can be written in this context :

$$(3.4.5) \qquad p\, u \,=\, (\gamma - 1)\, C_v\, Q\, T$$

and the boundary manifold $\mathcal{M}^{\rm jet}_{Q,T}$ is represented in velocity-pressure plane with an hyperbola, as shown on Figure 3.3. The resolution of the partial Riemann problem in space-time is easy : manifold $\mathcal{M}^{\rm jet}_{Q,T}$ is of codimension 1 and from a given state W_l in the vicinity of $\mathcal{M}^{\rm jet}_{Q,T}$, there exists a unique state $W_r \in \mathcal{M}^{\rm jet}_{Q,T}$, issued from W_l through a 1-wave. In the case proposed on Figure 3.4, this wave is a shock wave and the effect of the interaction at the boundary is the incoming of a shock wave inside the domain $\{x \leq L\}$.

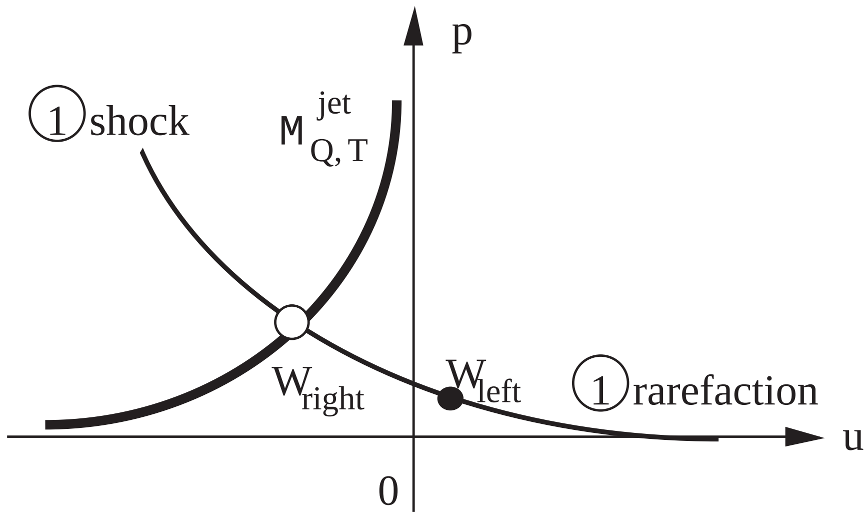

Figure 3.3 *Resolution in velocity-pressure plane of the subsonic jet inflow boundary condition associated with the manifold* $\mathcal{M}^{\rm jet}_{Q,T}$.

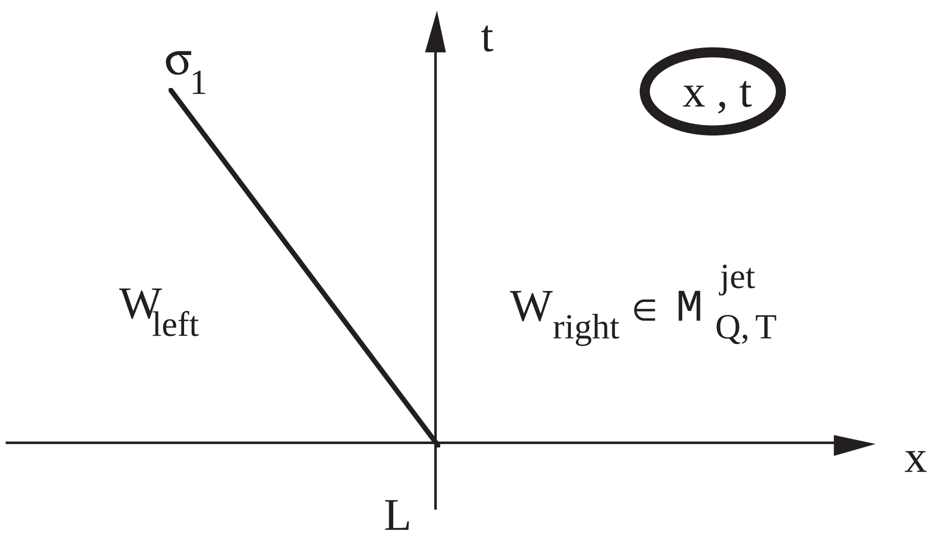

Figure 3.4 *Resolution in space-time of the subsonic jet inflow boundary condition associated with the manifold* $\mathcal{M}^{\rm jet}_{Q,T}$.

• **Subsonic nozzle inflow.** This case described on Figure 3.5. A given tank such the chamber of combustion of an engine to fix the ideas, contains some stagnation gas at rest at high temperature T_s and high pressure p_s. This gas is conducted to the inflow boundary through a compression wave in a convergent nozzle that maintains both total enthalpy H and specific entropy Σ. Recall that in a general manner, total enthalpy is defined according to

$$H \;\equiv\; \frac{1}{2} u^2 \,+\, \frac{\gamma\, p}{(\gamma - 1)\,\rho} \tag{3.4.6}$$

and specific entropy has been used at relation (1.1.12). When stagnation temperature T_s and stagnation pressure p_s are fixed, the total enthalpy H and the specific entropy Σ are given by

$$H \;=\; C_p\, T_s \,, \qquad \Sigma \;=\; C_v \,\mathrm{Log}\Big(\Big(\frac{T_s}{T_0}\Big)^{\gamma} \, \Big(\frac{p_s}{p_0}\Big)^{1-\gamma} \Big) \tag{3.4.7}$$

where T_0 is the temperature of reference state in relation (1.1.12). We define the associated boundary manifold $\mathcal{M}^{\mathrm{nozzle}}_{H,\Sigma}$ as the set of states such that total enthalpy and entropy are equal respectively to H and Σ ; we set

$$\mathcal{M}^{\mathrm{nozzle}}_{H,\Sigma} \;\equiv\; \left\{ \begin{array}{c} W = (\rho\,,\, q\,,\, \epsilon)^{\mathrm{t}} \in \Omega\,, \quad q \,=\, \rho\, u\,, \\ \epsilon \,=\, \frac{1}{2} \rho\, u^2 \,+\, \rho\, e\,, \quad p \,=\, (\gamma-1)\,\rho\, e\,, \\ \dfrac{1}{2} u^2 \,+\, \dfrac{\gamma\, p}{(\gamma - 1)\,\rho} \,=\, H\,, \quad \mathrm{Log}\Big(\dfrac{p\,\rho_0^{\gamma}}{p_0\,\rho^{\gamma}}\Big) \,=\, \dfrac{\Sigma}{C_v} \end{array} \right\} . \tag{3.4.8}$$

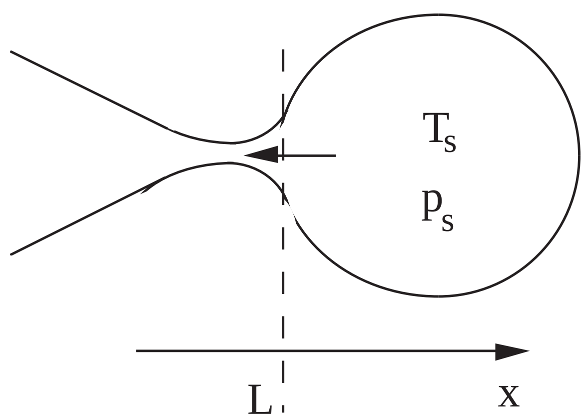

Figure 3.5 *Subsonic nozzle inflow at the boundary of the domain $x2L$. It is described by a manifold $\mathcal{M}^{\mathrm{nozzle}}_{H,\Sigma}$ parameterized by total enthalpy H and specific entropy Σ and in an equivalent way by stagnation temperature T_s and stagnation pressure p_s.*

The "projection" of manifold $\mathcal{M}^{\mathrm{nozzle}}_{H,\Sigma}$ in velocity-pressure plane is obtained by elimination of density ρ inside the relations presented at definition (3.4.8). It comes

$$\frac{1}{2}\, u^2 \,+\, \frac{\gamma}{\gamma - 1}\, \frac{p_0^{1/\gamma}}{\rho_0}\, \mathrm{e}^{\Sigma/(\gamma\, C_v)}\, p^{\frac{\gamma-1}{\gamma}} \;=\; H \tag{3.4.9}$$

and in terms of stagnation parameters :

$$\frac{u^2}{2\, C_p\, T_s} \,+\, \Big(\frac{p}{p_s}\Big)^{\frac{\gamma-1}{\gamma}} \;=\; 1\,. \tag{3.4.10}$$

The corresponding curve is represented on Figure 3.6 with the particular value of $\gamma = \frac{7}{5}$ *i.e.* $\frac{\gamma-1}{\gamma} = \frac{2}{7}$. We have also represented a graphical resolution of the partial Riemann problem $P(W_l,\, \mathcal{M}_{H,\Sigma}^{\mathrm{nozzle}})$ in the same plane.

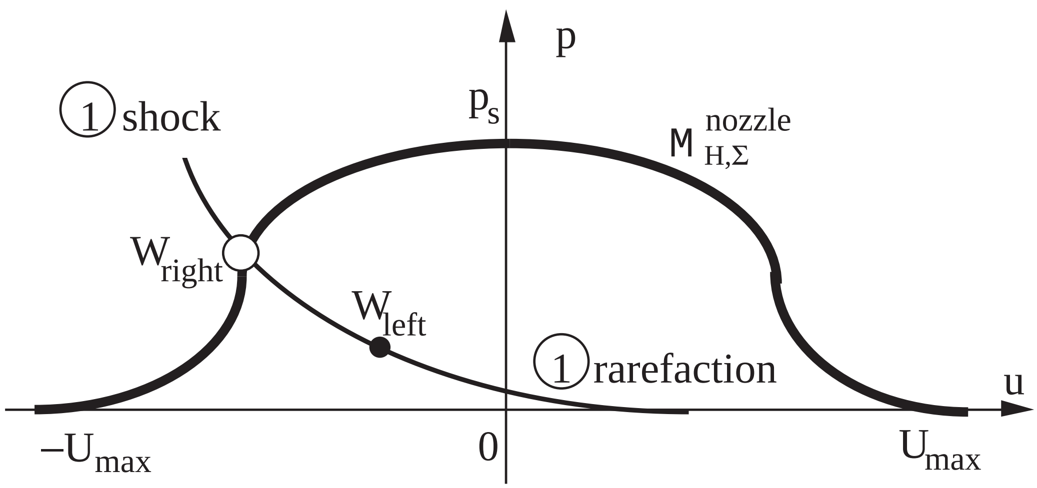

Figure 3.6 *Resolution in velocity-pressure plane of the subsonic nozzle inflow boundary condition associated with the manifold* $\mathcal{M}_{H,\Sigma}^{\mathrm{nozzle}}$.

• **Subsonic pressure outflow.** At a subsonic exit, it is classical to prescribe the static pressure. The linearized analysis shows that only one characteristic enters inside the domain of study and in consequence only one parameter is a priori necessary to close the boundary problem. The manifold associated to the pressure datum Π is denoted by $\mathcal{M}_{\Pi}^{\mathrm{pressure}}$ and is particularily simple to define :

$$\mathcal{M}_{\Pi}^{\mathrm{pressure}} \equiv \left\{ \begin{array}{c} W = \big(\rho\,,\, q = \rho\, u\,,\, \epsilon = \rho\, e + \dfrac{1}{2}\rho\, u^2\big)^{\mathrm{t}} \in \Omega, \\ (\gamma-1)\, \rho\, e \,=\, \Pi\,. \end{array} \right\} . \tag{3.4.11}$$

The resolution of the partial Riemann problem $P(W_l,\, \mathcal{M}_{\Pi}^{\mathrm{pressure}})$ is presented on Figure 3.7. Note here that an inflow of the fluid is absolutly compatible with the manifold $\mathcal{M}_{\Pi}^{\mathrm{pressure}}$.

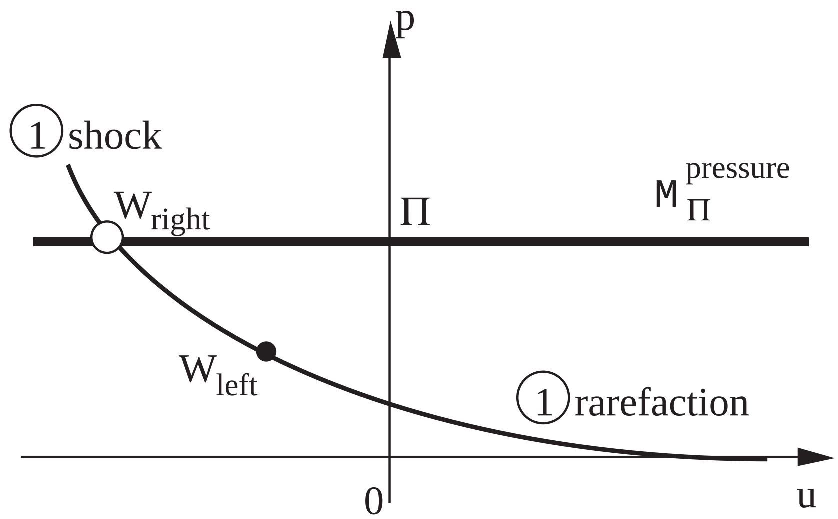

Figure 3.7 *Subsonic pressure outflow at the boundary of the domain x2L. It is described by a manifold* $\mathcal{M}_\Pi^{\mathrm{pressure}}$.

• **Supersonic outflow.** This case corresponds physically to natural boundary condition at the exit of a De Laval nozzle with sonic neck. No numerical datum has to be prescribed as we have seen previously with the linearized analysis. Nevertheless, the fact that the limit state corresponds to a supersonic outflow (*i.e.* $u \geq c$) can enter in conflict with the initial datum for example. In [Du87] and [Du88], we have analyzed numerically this kind of problem. A natural way to prescibe a "supersonic outflow" boundary condition is to introduce a boundary manifold $\mathcal{M}^{\mathrm{super}}$ that is now an half space as seen in section 2.5 :

(3.4.12) $$\mathcal{M}^{\mathrm{super}} \equiv \left\{ \begin{array}{c} W = \big(\rho\,,\, q = \rho\, u\,,\, \epsilon = \dfrac{\rho\, c^2}{\gamma\,(\gamma-1)} + \dfrac{1}{2} \rho\, u^2\big)^{\mathrm{t}} \in \Omega, \\ u - c \geq 0\,. \end{array} \right\}.$$

and to take into account the boundary condition with a partial Riemann problem $P(W_l,\, \mathcal{M}^{\mathrm{super}})$ as seen previously at Proposition 2.

3.5 Rigid wall and moving solid boundary.

• The first important case in practice is the surface of some rigid body. In the multidimensional case, if $\mathbf{n}$ is the external normal to the body and $\mathbf{u} = (u,\, v)$ the bidimensional field of velocity, this boundary condition takes the form of an impermeability condition :

(3.5.1) $\mathbf{u} \bullet \mathbf{n} \,=\, 0\,.$

At one space dimension, the external normal is equal to 1 and the condition (3.5.1) reduces to a nonlinear version of condition (3.2.11), *id est*

(3.5.2) $u \,=\, 0\,.$

As in previous cases, this physically given boundary condition can be incompatible with the state $W(L^-, t)$ present near the boundary. We propose here to write in a weak way the boundary condition (3.5.2) with a partial Riemann problem associated with boundary manifold $\mathcal{M}_r = \mathcal{M}_0^{\rm velocity}$:

(3.5.3) $\qquad \mathcal{M}_0^{\rm velocity} \equiv \Big\{ W = \big(\rho\,,\, q = \rho\, u\,,\, \epsilon\big)^{\rm t} \in \Omega, \quad u = 0 \Big\}.$

Proposition 3. Stationary state for a rigid wall.
Let $W_l \in \Omega$ be some given state, $\mathcal{M}_0^{\rm velocity}$ the manifold defined in (3.5.3) and $W(x,\, t) = U(\frac{x}{t}\,;\, W_l,\, \mathcal{M}_0^{\rm velocity})$ the entropy solution of the partial Riemann problem $P(W_l,\, \mathcal{M}_0^{\rm velocity})$ proposed at Theorem 2. Then the velocity u^* of the stationary state $U(0\,;\, W_l,\, \mathcal{M}_0^{\rm velocity})$ is null and we have

(3.5.4) $\qquad U(0\,;\, W_l,\, \mathcal{M}_0^{\rm velocity}) = \big(\rho^*\,,\, 0\,,\, \dfrac{p^*}{\gamma - 1}\big)^{\rm t}.$

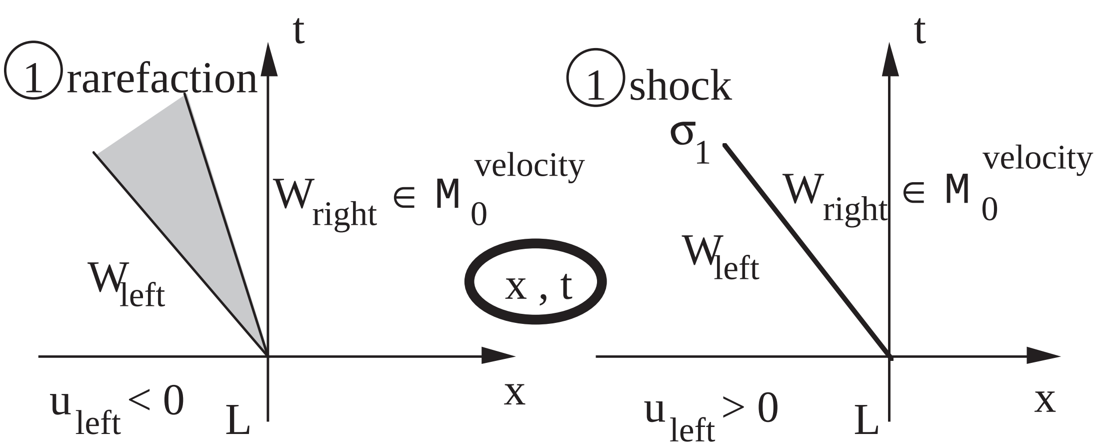

Figure 3.8 *Rigid wall boundary condition. The velocity of the stationary state is equal to zero.*

• The proof of Proposition 3 is clear if velocity u_l of left state W_l is negative. In this case, the 1-wave that links state W_l and manifold $\mathcal{M}_0^{\rm velocity}$ is a rarefaction and the fan $]\mu_1^-\,,\, \mu_1^+[$ in space-time plane is defined by the conditions

(3.5.5) $\qquad u_l - c_l \equiv \mu_1^- < \xi < \mu_1^+ \equiv u_r - c_r = -c_r < 0$

as presented on Figure 3.8. Then we have

(3.5.6) $\qquad U(0\,;\, W_l,\, \mathcal{M}_0^{\rm velocity}) = W_r$

and property (3.5.4) is true in this particular case. If $u_l > 0$, the 1-wave is a shock, we have from (1.6.14) and (1.4.16) and due to the classical inequality $p^* \geq p_l$:

$$\sigma_1 \;\equiv\; u_l - \sqrt{\frac{p^* \,+\, \mu^2\, p_l}{(1-\mu^2)\,\rho_l}} \;\leq\; u_l - \sqrt{\frac{1-\mu^2}{\rho_l\,(p^* + \mu^2\, p_l)}}\,\big(p - p_l\big) \;\equiv\; u^*\,. \tag{3.5.7}$$

Property (3.5.4) is a consequence of the general order (3.5.7) for the different waves : the celerity σ_1 of the 1-wave is less or equal than the celerity u^* of the 2-wave. For the partial Riemann problem $P(W_l,\, \mathcal{M}_0^{\text{velocity}})$, we have $u^* = 0$ and Proposition 3 is established. $\square$

• A consequence of Proposition 3 is the fact that the only wave of partial Riemann problem $P(W_l,\, \mathcal{M}_0^{\text{velocity}})$ is located in the quarter of space $\{x < L,\; t > 0\}$ as depicted on Figure 3.8. We remark also that in this quarter of space, the solution of the partial Riemann problem $P(W_l,\, \mathcal{M}_0^{\text{velocity}})$ is **identical** to the solution of the Riemann problem $R(W_l,\, \widetilde{W_l})$ between left state W_l and its "mirror" $\widetilde{W_l}$. Recall that, following *e.g.* Roache [Ro72], the mirror state is defined by the conditions

$$\widetilde{\rho_l} \;=\; \rho_l\,, \qquad \widetilde{u_l} \;=\; -u_l\,, \qquad \widetilde{p_l} \;=\; p_l\,. \tag{3.5.8}$$

One advantage of the notion of partial Riemamm problem is that we do not need to use this very numerical notion in what follows.

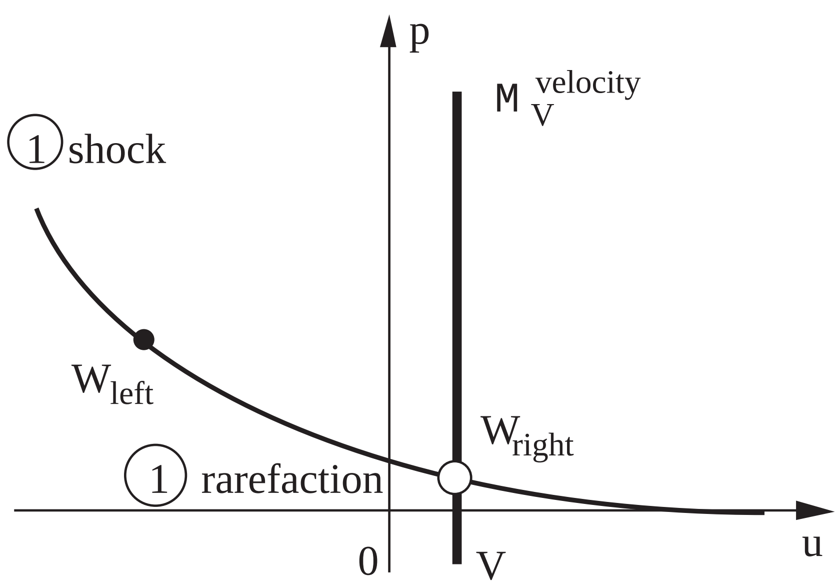

Figure 3.9 *Moving boundary condition at the boundary of the domain* $\{x \leq L\}$*. It is described by a manifold* $\mathcal{M}_V^{\text{velocity}}$.

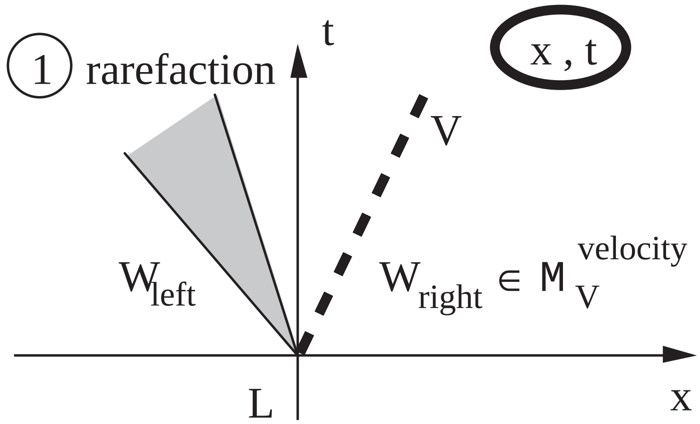

Figure 3.10 *Solution in space-time of a moving boundary condition at the boundary with imposed velocity V.*

• The next example is motivated by fluid-structure interaction problems. In this kind of configuration, the boundary of the fluid domain is a **moving boundary** and the displacements are sufficiently small compared to typical length scales of the problem and can be neglected. Nevertheless, the boundary is moving and the value V of velocity is supposed to be given. We consider the weak treatment of the boundary condition $u = V$ by the introduction of the manifold $\mathcal{M}_V^{\text{velocity}}$ defined as a generalization of (3.5.3) :

(3.5.9) $$\mathcal{M}_V^{\text{velocity}} \equiv \Big\{ W = \big(\rho \,,\, q = \rho\, u \,,\, \epsilon\big)^{\text{t}} \in \Omega, \quad u = V \Big\} .$$

The resolution of the partial Riemann problem $P(W_l,\, \mathcal{M}_V^{\text{velocity}})$ at the boundary is straightforward (see [Du99]). The unknown pressure $p^*(V)$ on the boundary is solution of the following equations introduced in (1.6.6) :

(3.5.10) $$\begin{cases} V \,-\, u_l \,+\, \psi(p^*(V) \,;\, \rho_l \,,\, p_l \,;\, \gamma) \,=\, 0 \,, & p^*(V) \,<\, p_l \\ V \,-\, u_l \,+\, \varphi(p^*(V) \,;\, \rho_l \,,\, p_l \,;\, \gamma) \,=\, 0 \,, & p^*(V) \,>\, p_l \,, \end{cases}$$

that can be explicited in this case :

(3.5.11) $$\begin{cases} V \,-\, u_l \,+\, \dfrac{2\, c_l}{\gamma - 1} \Big[\Big(\dfrac{p^*(V)}{p_l} \Big)^{\frac{\gamma-1}{2\,\gamma}} - 1 \Big] \,=\, 0 & \text{if } \; V \,-\, u_l \,\geq\, 0 \\ V \,-\, u_l \,+\, \dfrac{\sqrt{2}\,\big(p^*(V) - p_l\big)}{\sqrt{\rho_l \big[(\gamma+1)\, p^*(V) + (\gamma-1)\, p_l \big]}} \,=\, 0 & \text{if } \; V \,-\, u_l \,\leq\, 0 \,. \end{cases}$$

With this approach, it is very easy to precise the difference $p^*(V) - p^*(0)$ of boundary pressures associated with moving boundary with velocity V and a rigid wall. At the first order, we find :

(3.5.12) $$p^*(V) \,=\, p^*(0) \,-\, \rho_l\, c_l\, V \,+\, O(V^2) \,.$$

We have developed in [Du99] a complete procedure to take into account numerically a displacement of the boundary of small amplitude and small celerity with

the so-called **limiting flux for moving boundary**.

4) Application to the finite volume method.

4.1 **Godunov finite volume method.**

• The system of conservation laws

(4.1.1) $$\frac{\partial}{\partial t} W(x,\,t) + \frac{\partial}{\partial x} F(W(x,\,t)) \;=\; 0$$

is now discretized with the finite volume method. We consider a finite one-dimensional domain of study $]0,\,L[$ and we divide it with a mesh $\mathcal{T}$ initially composed by a family $\mathcal{S}_{\mathcal{T}}$ of $J+1$ vertices (note that integer J depends on the mesh $\mathcal{T}$) :

(4.1.2) $$\mathcal{S}_{\mathcal{T}} \;=\; \big\{\, 0 = x_0 \,<\, x_1 \,<\cdots<\, x_j \,<\, x_{j+1} \,<\cdots<\, x_J = L \,\big\}\,.$$

We construct from vertices $(x_j)_{0\leq j\leq J}$ the family $\mathcal{E}_{\mathcal{T}}$ of finite elements or control volume cells K :

(4.1.3) $$]0,\,L[\;=\; \bigcup_{K\in\mathcal{E}_{\mathcal{T}}} \overline{K}\,, \quad K =]x_j\,,\,x_{j+1}[\;\in \mathcal{E}_{\mathcal{T}}\,, \quad 0\leq j\leq J-1\,,$$

we remark that two different cells K_1 and K_2 have an intersection with null measure :

(4.1.4) $$\mathrm{mes}\,(K_1\cap K_2) \;=\; 0\,, \qquad K_1 \neq K_2 \;\in \mathcal{E}_{\mathcal{T}}$$

and we introduce the two vertices $S_-(K)$ and $S_+(K)$ that define the boundary ∂K of element K :

(4.1.5) $$K \;=\;]S_-(K)\,,\,S_+(K)[\,, \qquad \partial K \;=\; \big\{\, S_-(K)\,,\,S_+(K)\,\big\}\,, \quad K\in\mathcal{E}_{\mathcal{T}}\,.$$

• For each element K of the mesh $(K\in\mathcal{E}_{\mathcal{T}})$, we introduce the **mean value** W_K of the solution of conservation law (4.1.1) :

(4.1.6) $$W_K \;=\; \frac{1}{\mid K\mid}\int_K W(x)\,\mathrm{d}x\,, \qquad K\in\mathcal{E}_{\mathcal{T}}$$

and we consider the ordinary differential equation satisfied by the functions $[0,+\infty[\, \ni t \longmapsto W_K(t)\in\Omega$. We integrate the conservation law (4.1.1) in space in each control volume K. After integrating by parts the divergent term $\frac{\partial F(W)}{\partial x}$, it comes

(4.1.7) $$|K|\,\frac{\mathrm{d}W_K}{\mathrm{d}t} + F\Big(W\big(S^+(K),\,t\big)\Big) - F\Big(W\big(S^-(K),\,t\big)\Big) \;=\; 0\,, \quad K\in\mathcal{E}_{\mathcal{T}}\,.$$

The expression (4.1.7) shows that a discretization procedure can be achieved if we are able to define the flux F_S for each vertex $S\in\mathcal{S}_{\mathcal{T}}$ in terms of the data, *i.e.* the mean values $\{W_K\,,\,K\in\mathcal{E}_{\mathcal{T}}\}$ and of the boundary conditions. This **numerical modelling** characterizes the so-called finite volume method (see *e.g.* Godunov *et al* [GZIKP79], Patankar [Pa80], Harten, Lax and Van Leer [HLV83]

or Faille, Gallouët and Herbin [FGH91] among others) and we will precise some efficient choices in practice in what follows.

• We restrict ourselves for a time to vertices $S \in \mathcal{S}_{\mathcal{T}}$ that are such that **two** finite elements $K_l(S)$ and $K_r(S)$ possess the vertex S in their boundary. It is the case when vertex S is internal to the domain $]0, L[$:

(4.1.8) $\qquad \partial K_l(S) \cap \partial K_r(S) = \{S\}, \qquad S \in \mathcal{S}_{\mathcal{T}}$ not located on the boundary.

In practice, S is one of the vertices $x_1, \cdots, x_{J-1}$ and $K_l(x_j) =]x_{j-1}, x_j[$, $K_r(x_j) =]x_j, x_{j+1}[$. With Godunov [Go59] we propose a numerical model for the numerical flux F_S at vertex S with the help of the Riemann problem. We denote by $U(\xi \equiv \frac{x}{t}; W_l, W_r)$ the entropy solution of the Riemann problem $R(W_l, W_r)$. We set for internal vertices

$$F_S = F\Big(U\big(0; W_{K_l(S)}, W_{K_r(S)}\big)\Big), \qquad S \in \{x_1, \cdots, x_{J-1}\} \tag{4.1.9}$$

and by this way, we have constructed an ordinary differential equation

$$\frac{\mathrm{d}W_K}{\mathrm{d}t} + \frac{1}{|K|}\Big(F_{S_+(K)} - F_{S_-(K)}\Big) = 0, \qquad K \in \mathcal{E}_{\mathcal{T}} \tag{4.1.10}$$

for control volumes K such that their boundary ∂K is not located on the boundary.

4.2 Boundary fluxes.

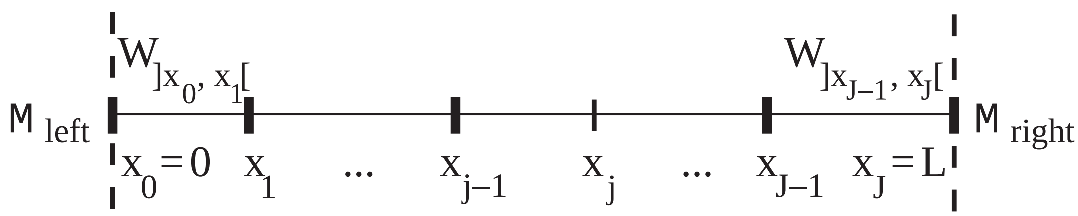

Figure 4.1 *Flux boundary conditions for the finite volume method. At the two boundaries of this unidimensional domain, the partial Riemann problems have to be set to compute the flux at the boundary.*

• We wish now to give a sense to equation (4.1.10) for all control volumes K of the mesh $\mathcal{T}$. We make the hypothesis that at the left interface $\{x = 0\}$ and at the right interface $\{x = L\}$ of the domain, a left boundary manifold $\mathcal{M}_l$ and a right boundary manifold $\mathcal{M}_r$ are determined by the physical data (see Figure 4.1). Each boundary manifold is supposed to be chosen inside the family proposed previously at relations (3.4.1) for given state W_∞, (3.4.2) for subsonic jet inflow $\mathcal{M}^{\text{jet}}_{Q,T}$ (with mass flux parameter Q and temperature T), (3.4.8) for subsonic nozzle inflow $\mathcal{M}^{\text{nozzle}}_{H,\Sigma}$ (with numerical datum defined with total enthalpy H and specific entropy Σ), (3.4.11) for subsonic pressure outflow

$\mathcal{M}_\Pi^{\text{pressure}}$ (with given static pressure Π), (3.4.12) for supersonic outflow $\mathcal{M}^{\text{super}}$ or (3.5.9) for a moving boundary condition $\mathcal{M}_V^{\text{velocity}}$ (with celerity V) :

$$(4.2.1) \qquad \mathcal{M}_l\,,\,\mathcal{M}_r \,\in\, \big\{\{W_\infty\}\,,\,\mathcal{M}_{Q,T}^{\text{jet}}\,,\,\mathcal{M}_{H,\Sigma}^{\text{nozzle}}\,,\,\mathcal{M}_\Pi^{\text{pressure}}\,,\,\mathcal{M}^{\text{super}}\,,\,\mathcal{M}_V^{\text{velocity}}\big\}\,.$$

• We denote by $U(\xi \equiv \frac{x}{t}\,;\,\mathcal{M}_l\,,\,W_r)$ and $U(\xi \equiv \frac{x}{t}\,;\,W_l\,,\,\mathcal{M}_r)$ the selfsimilar entropy solution of the partial Riemann problems $P(\mathcal{M}_l\,,\,W_r)$ and $P(W_l\,,\,\mathcal{M}_r)$ proposed at Theorem 2 and relation (2.4.14). The boundary flux F_{x_0} at the left interface is evaluated with the resolution of the partial Riemann problem $P(\mathcal{M}_l\,,\,W_{]x_0,\,x_1[})$ between the left boundary datum $\mathcal{M}_l$ and the state value W_r in the first cell $]x_0,\,x_1[$:

$$(4.2.2) \qquad F_{x_0} \;=\; F\Big(U\big(0\,;\,\mathcal{M}_l\,,\,W_{]x_0,\,x_1[}\big)\Big)\,.$$

In a similar way, the boundary flux F_{x_J} at the right interface $\{x = L\}$ is obtained thanks to the resolution of the partial Riemann problem $P(W_{]x_{J-1},\,x_J[}\,,\,\mathcal{M}_r)$ between the left state W_l in the last cell $]x_{J-1},\,x_J[$ and the right boundary datum $\mathcal{M}_r$:

$$(4.2.3) \qquad F_{x_J} \;=\; F\Big(U\big(0\,;\,W_{]x_{J-1},\,x_J[}\,,\,\mathcal{M}_r\big)\Big)\,.$$

With this choice of boundary fluxes F_{x_0} and F_{x_J}, the ordinary differential equation (4.1.10) takes a mathematical sense for all the control volumes $K \in \mathcal{E}_{\mathcal{T}}$.

4.3 Strong nonlinearity at the boundary.

• We focus here on the fact that the introduction of a partial Riemann problem allows the treatment of strongly nonlinear effects at the boundary. Consider to fix the ideas the boundary $\{x = L\}$ associated with computational domain $]0,\,L[$. We suppose that the physical conditions at this boundary are taken into account with the help of some manifold $\mathcal{M}_r$. In order to consider weakly the boundary condition, we have introduced in (3.3.5) the set $\beta(\mathcal{M}_r)$ of admissible values at the boundary :

$$(4.3.1) \qquad \beta\big(\mathcal{M}_r\big) \;=\; \left\{ \begin{array}{l} \text{values at } \frac{x}{t} = 0^- \text{ of the entropic solution of the} \\ \text{partial Riemann problem } P(W,\,\mathcal{M}_r),\, W \in \Omega \end{array} \right\}.$$

Then the boundary condition at $x = L$ is considered in the continuous case in (3.3.4) and we have by definition :

$$(4.3.2) \qquad W(L^-,\,t) \,\in\, \beta\big(\mathcal{M}_r\big)\,.$$

This definition is compatible with the proposed implementation with the finite volume method : the boundary flux evaluated in (4.2.3) is the flux of the particular state at $\xi = 0$ in the selfsimilar solution $U(\xi\,;\,W_l,\,\mathcal{M}_r)$ of the partial Riemann problem $P(W_l,\,\mathcal{M}_r)$.

• In lot of cases, the boundary condition acts as in the linear regime, *i.e.* the left state W_l in the partial Riemann problem $P(W_l,\,\mathcal{M}_r)$ lies in the

vicinity of the manifold $\mathcal{M}_r$ and the stationary value $U(\xi = 0\,;\, W_l,\, \mathcal{M}_r)$ is located **inside** the boundary manifold $\mathcal{M}_r$:

(4.3.3) $\qquad U(\xi = 0\,;\, W_l,\, \mathcal{M}_r) \,\in \mathcal{M}_r \quad : \quad$ weak nonlinearity at the boundary.

This case occurs typically for a manifold of codimension 1. Consider to fix the ideas a manifold $\mathcal{M}_{\Pi}^{\text{pressure}}$ where the static pressure is fixed by the physical outflow conditions. The linearized approach in the vicinity of the (unknown) state $W^* = U(\xi = 0\,;\, W_l,\, \mathcal{M}_r)$ is in general correct (see Figure 4.2) and the classical approach (see *e.g.* Viviand and Veuillot [VV78], Chakravarthy [Ch83], Osher and Chakravarthy [OC83]) consists in determining a state W^* at the boundary that satifies strongly the boundary condition, *i.e.* satisfies the condition $W^* \in \mathcal{M}_r$ and satisfies also the linearized gas dynamics equations written under a characteristic form along the two characteristics going outside the domain of study.

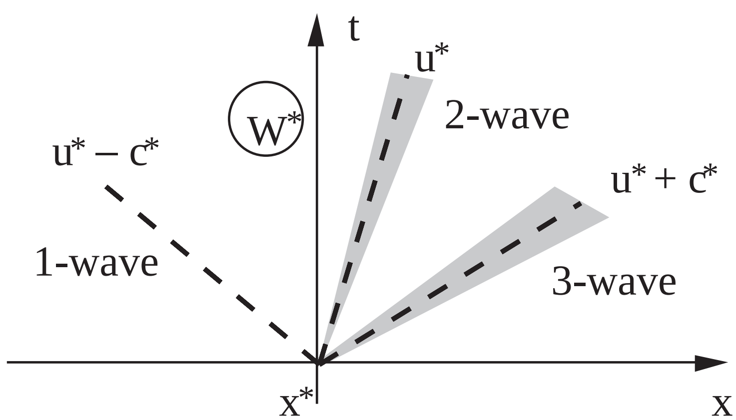

Figure 4.2 *Weakly nonlinear boundary condition. The state W* at the boundary x=x* belongs to the boundary manifold.*

• This approach is in defect when the limit state is far from the boundary manifold $\mathcal{M}_r$. By example, a supersonic boundary condition is prescribed but the limiting state $W(L^-, t)$ corresponds to a subsonic outflow. On the contrary, a subsonic outflow with pressure Π is supposed to be given but the limiting state $W(L^-, t)$ is associated with a supersonic exit ! The flexibility of the partial Riemann problem allows the treatment with strong nonlinear waves. When condition (4.3.3) is in defect, the boundary condition (4.2.2) or (4.2.3) takes into account the physical data that constructs the interaction at the boundary whereas these data are not directly used for the final computation of the boundary flux (4.2.2) or (4.2.3).

4.4 **Extension to second order accuracy and to two space dimensions.**

• We detail in this section a generalization for unstructured meshes of the "Multidimensional Upwindcentered Scheme for Conservation Laws" proposed by Van Leer [VL79]. At one space dimension on a uniform mesh, it is classical to consider a scalar field z among the primitive variables, *i.e.*

$$z \in \{ \rho , u , v , p \} \qquad \text{(primitive variables)} \tag{4.4.1}$$

and instead of computing the interface flux with relation (4.1.9), to first construct two interface states W_S^- and W_S^+ on each side of the interface S. Then the flux is evaluated by the decomposition of the discontinuity :

$$F_S = F\Big(U\big(0\,;\, W_S^- , W_S^+\big)\Big) , \qquad S \in \{ x_1, \cdots, x_{J-1} \} . \tag{4.4.2}$$

This nonlinear interpolation is done with a so-called "slope limiter" $\varphi(\bullet)$ that operates on each variable proposed in (4.4.1) and we have typically when a left-right invariance is assumed [Du91] :

$$z_S^- = z_{j-1/2} + \frac{1}{2}\,\varphi\bigg(\frac{z_{j-1/2} - z_{j-3/2}}{z_{j+1/2} - z_{j-1/2}}\bigg)\big(z_{j+1/2} - z_{j-1/2}\big) , \qquad S = x_j \tag{4.4.3}$$

$$z_S^+ = z_{j+1/2} - \frac{1}{2}\,\varphi\bigg(\frac{z_{j+3/2} - z_{j+1/2}}{z_{j+1/2} - z_{j-1/2}}\bigg)\big(z_{j+1/2} - z_{j-1/2}\big) , \qquad S = x_j . \tag{4.4.4}$$

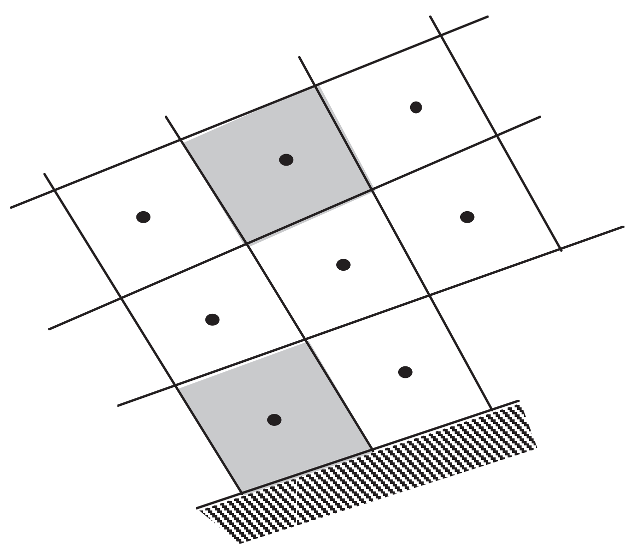

Figure 4.3 *Structured cartesian mesh. The control volumes are exactly the elements of mesh $\mathcal{T}$.*

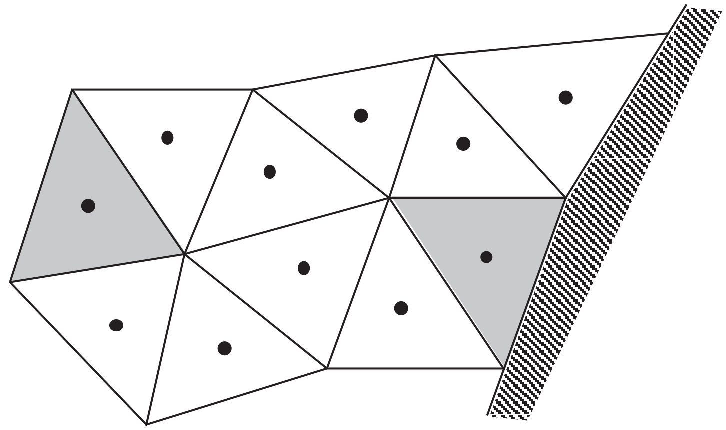

Figure 4.4 *Unstructured mesh composed by triangular elements. The control volumes are exactly the elements of mesh $\mathcal{T}$.*

• We focus now on two problems for the extension to second order accuracy of Godunov finite volume method. One is associated with the use of unstructured meshes and the other with the treatment of boundary conditions. As in the one-dimensional case, the domain of study is decomposed into finite elements (or control volumes) $K \in \mathcal{E}_{\mathcal{T}}$ than can be structured in a cartesian way (Figure 4.3) or with a cellular complex as in Figure 4.4. In both cases, the intersection of two finite elements define an interface $f \in \mathcal{F}_{\mathcal{T}}$. We denote by $\mathbf{n}_f$ the normal at the interface f that separates a left control volume $K_l(f)$ and a right control volume $K_r(f)$. The ordinary differential equation (4.1.7) is replaced by a multidimensional version :

$$(4.4.5) \qquad |K| \, \frac{\mathrm{d}W_K}{\mathrm{d}t} + \sum_{f \subset \partial K} |f| \, \Phi\big(W_K \,,\, \mathbf{n}_f \,,\, W_{K_r(f)}\big) \,=\, 0 \,, \qquad K \in \mathcal{E}_{\mathcal{T}} \,.$$

For internal interfaces, the function $\Phi\big(\bullet \,,\, \mathbf{n}_f \,,\, \bullet\big)$ is equal to the flux of the solution at $\frac{x}{t} = 0$ of the Riemann problem between states $W_{K_l(f)}$ and $W_{K_r(f)}$ in the one-dimensional direction along normal $\mathbf{n}_f$ in order to take into account the invariance by rotation of the equations of gas dynamics (see [GR96]). We suppose also that for each interface f of the boundary, a boundary manifold $\mathcal{M}_f$ of codimension $p(f)$ is given and the normal direction $\mathbf{n}_f$ is by convention external to the domain of study. In consequence, the control cell $K = K_l(f)$ has the face f in its boundary $(f \subset \partial(K_l(f)))$ and in relation (4.4.5), the state $W_{K_r(f)}$ belongs to $\mathcal{M}_f$ $(W_{K_r(f)} \in \mathcal{M}_f)$ and is equal to the state $W^{p(f)}$ introduced in relation (2.4.2) when solving the partial Riemann problem $P(W_{K_l(f)} \,,\, \mathbf{n}_f \,,\, \mathcal{M}_f)$ between state $W_{K_l(f)}$ and manifold $\mathcal{M}_f$.

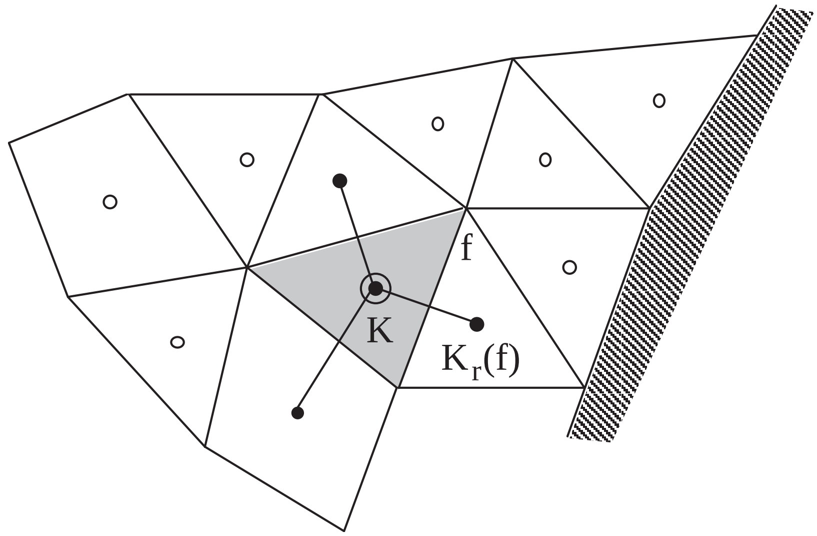

Figure 4.5 *Cellular complex mesh with triangles and quadrangles. Three neighbouring cells are necessary to determine the gradient in triangle K and to limit eventually its variation.*

• We consider now a finite element K internal to the domain. The extension to second order accuracy of the finite volume scheme consists in replacing the arguments $W_{K_l(f)}$ and $W_{K_r(f)}$ in relation (4.4.5) by nonlinear extrapolations $W_{K_l(f),\,f}$ and $W_{K_r(f),\,f}$ on each side of the boundary of state data and evaluated as described in what follows. We first introduce the set $\mathcal{N}(K)$ of neighbouring cells of given finite element $K \in \mathcal{E}_{\mathcal{T}}$, as illustrated on Figure 4.5 :

(4.4.6) $$\mathcal{N}(K) \;=\; \big\{ L \in \mathcal{E}_{\mathcal{T}}, \quad \exists\, f \in \mathcal{F}_{\mathcal{T}}, \quad f \subset \partial K \cap \partial L \big\} .$$

For $L \in \mathcal{N}(K)$, we suppose by convention that the normal $\mathbf{n}_f$ to the face $f \subset \partial K \cap \partial L$ is external to the element K *id est* $K_r(f) = K$, $K_l(f) = L$. We introduce also the point $y_{K,\,f}$ on the interface $f \subset \partial K$ that links the barycenters x_K and $x_{K_r(f)}$:

(4.4.7) $$\left\{ \begin{array}{l} y_{K,\,f} \;\equiv\; (1-\theta_{K,\,f})\, x_K \;+\; \theta_{K,\,f}\, x_{K_r(f)} \,, \qquad y_{K,\,f} \in f \,, \\ \qquad\qquad f \subset \partial K \,, \qquad K \text{ finite element internal to mesh } \mathcal{T} . \end{array} \right.$$

Then, following Pollet [Po88], for z equal to one scalar variable of the family :

(4.4.8) $$z \;\in\; \{ \rho \,,\, \rho\, u \,,\, \rho\, v \,,\, p \}$$

we evaluate a mean value $\overline{z_{K,\,f}}$ on the interface f :

(4.4.9) $$\overline{z_{K,\,f}} \;=\; (1-\theta_{K,\,f})\, z_K \;+\; \theta_{K,\,f}\, z_{K_r(f)}$$

and the gradient $\nabla z(K)$ of field $z(\bullet)$ in volume K with a Green formula :

(4.4.10) $$\nabla z(K) \;=\; \frac{1}{\mid K \mid} \int_{\partial K} \overline{z}\, \mathbf{n}\, \mathrm{d}\gamma \;=\; \frac{1}{\mid K \mid} \sum_{f \subset \partial K} \mid f \mid\, \overline{z_{K,\,f}}\;\, \mathbf{n}_f \,,\;\; K \in \mathcal{E}_{\mathcal{T}} \,.$$

• An ideal extrapolation of field $z(\bullet)$ at the interface f would be :

(4.4.11) $$z_{K,\,f} \;=\; z_K \,+\, \nabla z(K) \bullet \big(y_{K,\,f} - x_K \big)$$

but the corresponding scheme is unstable as seen by Van Leer [VL77]. When the variation $\nabla z(K) \bullet \big(y_{K,\,f} - x_K \big)$ is too important, it has to be "limited" as first suggested by Van Leer [VL77]. For doing this in a very general way, we introduce the minimum $m_K(z)$ and the maximum $M_K(z)$ of field $z(\bullet)$ in the neighbouring cells :

(4.4.12) $$m_K(z) \;=\; \min \big\{ \, z_L \,, \quad L \in \mathcal{N}(K) \, \big\}$$

(4.4.13) $$M_K(z) \;=\; \max \big\{ \, z_L \,, \quad L \in \mathcal{N}(K) \, \big\} \,.$$

If the value z_K is extremum among the neighbouring ones, *i.e.* if $z_K \,\leq\, m_K(z)$, or $z_K \,\geq\, M_K(z)$, we impose that the interpolated value $z_{K,\,f}$ is equal to the cell value z_K :

(4.4.14) $$z_{K,\,f} \;=\; z_K \qquad \text{if} \quad z_K \,\leq\, m_K(z) \quad \text{or} \quad z_K \,\geq\, M_K(z) \,, \quad f \subset \partial K \,.$$

When on the contrary z_K lies inside the interval $[m_K(z),\, M_K(z)]$, we impose that the variation $z_{K,\,f} - z_K$ is limited by some coefficient $k \, (0 \,\leq\, k \,\leq\, 1)$ of the variations $z_K - m_K(z)$ and $M_K(z) - z_K$. We introduce a nonlinear extrapolation of the field $z(\bullet)$ between center x_K and boundary face $y_{K,\,f}$ $(f \subset \partial K)$:

(4.4.15) $$z_{K,\,f} \;=\; z_K \,+\, \alpha_K(z) \, \nabla z(K) \bullet \big(y_{K,\,f} - x_K \big) \,, \qquad f \subset \partial K$$

with a limiting coefficient $\alpha_K(z)$ satisfying the following conditions :

(4.4.16) $$\left\{ \begin{array}{l} 0 \,\leq\, \alpha_K(z) \,\leq\, 1 \,, \quad z(\bullet) \ \text{scalar field defined in (4.4.8)}, \quad K \in \mathcal{E}_{\mathcal{T}} \\ k \, (z_K - m_K(z)) \,\leq\, \alpha_K(z) \, \nabla z(K) \bullet \big(y_{K,\,f} - x_K \big) \,\leq\, k \, (M_K(z) - z_K) \\ \hfill \forall \, f \subset \partial K \,, \quad K \in \mathcal{E}_{\mathcal{T}} \,. \end{array} \right.$$

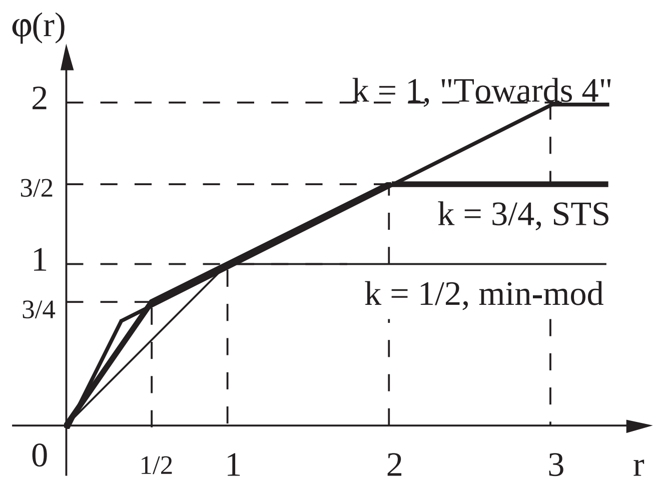

Figure 4.6 *Examples of limiter functions that can be easily extended to unstructured meshes.*

Then $\alpha_K(z)$ is chosen as big as possible and inferior or equal to 1 in order to satisfy the constraints (4.4.16) as displayed on Figure 4.6 :

$$\alpha_K(z) \;=\; \min \left[1 \,,\, k \, \frac{\min\big(M_K(z) - z_K \,,\, z_K - m_K(z) \big)}{\max \big\{ \,|\, \nabla z(K) \bullet (y_{K,\,f} - x_K) \,|\,,\; f \subset \partial K \,\big\}} \right] . \mathrm{v} \qquad (4.4.17)$$

• In the one dimensional case with a regular mesh, it is an exercice to re-write the extrapolation (4.4.15) under the usual form (4.4.3) in the context of finite differences. In this particular case, some limiter functions $r \longmapsto \varphi_k(r)$ associated with particular parameters k are shown on Figure 4.6. For $k = 1$, we recover the initial limiter proposed by Van Leer in the fourth paper of the family "Towards the ultimate finite difference scheme..." [VL77] ; for this reason, we have named it the "Towards 4" limiter (see Figure 4.6). When $k = \frac{1}{2}$ we obtain the "min-mod" limiter proposed by Harten [Ha83]. The intermediate value $k = \frac{3}{4}$ is a good compromise between the "nearly unstable" choice $k = 1$ and the "too compressive" min-mod choice. We have named it STS and it has been chosen for our Euler computations in [DM92].

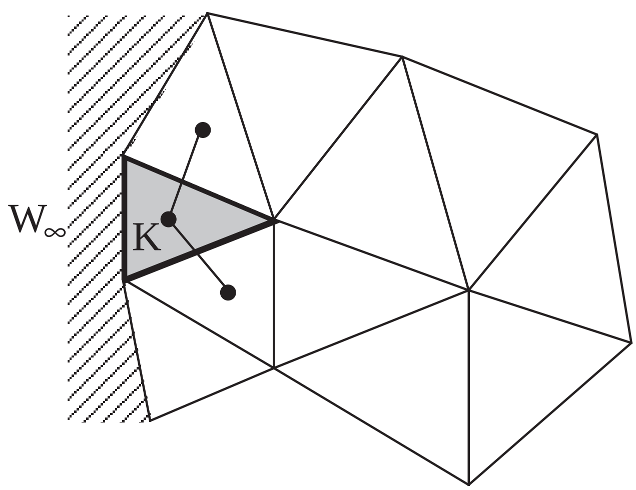

Figure 4.7 *Slope limitation at a fluid boundary.*

• We explain now the way the preceeding scheme is adapted near the boundary. We first consider a **fluid boundary**. When K is a finite element with some face $g \subset \partial K$ lying on the boundary, we still define the set $\mathcal{N}(K)$ of neighbouring cells by the relation (4.4.6) as shown on Figure 4.7. The number of neighbouring cells is just less important in this case. Then points $y_{K,\,f}$ are introduced by relation (4.4.7) if face f does not lie on the boundary and by taking the barycenter of face g if it is lying on the boundary. The only difference is the way the values $\overline{z_{K,\,g}}$ are extrapolated for the face g that is on the boundary ; we set

(4.4.18) $\overline{z_{K,\,g}} \;=\; z_K \;,\quad g \subset \partial K \;,\quad g$ face lying on the boundary of the domain.

When values $\overline{z_{K,\,f}}$ are determined for all the faces $f \subset \partial K$, the gradient $\nabla z(K)$, the minimal $m_K(z)$ and maximal $M_K(z)$ values among the neighbouring cells are still determined with the relations (4.4.10), (4.4.12) and (4.4.13) respectively. The constraints (4.4.16) remain unchanged except that no limitation process is due to the faces lying on the boundary. In a precise way, we set :

$$(4.4.19)\ \alpha_K(z) = \min\Big[1, \frac{k \quad \min\big(M_K(z) - z_K\,,\, z_K - m_K(z)\big)}{\max\big\{|\nabla z(K) \bullet (y_{K,\,f} - x_K)|, f \subset \partial K, K_r(f) \in \mathcal{N}(K)\big\}}\Big].$$

Then the interpolated values $z_{K,\,f}$ for all the faces $f \subset \partial K$ are again predicted with the help of relation (4.4.15).

• For a **rigid wall**, the limitation process is a little modified, as presented at Figure 4.8. First we introduce the limit face g inside the set of neighbouring cells :

$$(4.4.20) \qquad \left\{ \begin{array}{l} \mathcal{N}(K) = \big\{ L \in \mathcal{E}_{\mathcal{T}},\ \exists f \subset \partial K \cap \partial L \big\} \cup \\ \qquad\qquad \cup \big\{ g \in \mathcal{F}_{\mathcal{T}},\ g \subset \partial K,\ g \text{ on the boundary} \big\}. \end{array} \right.$$

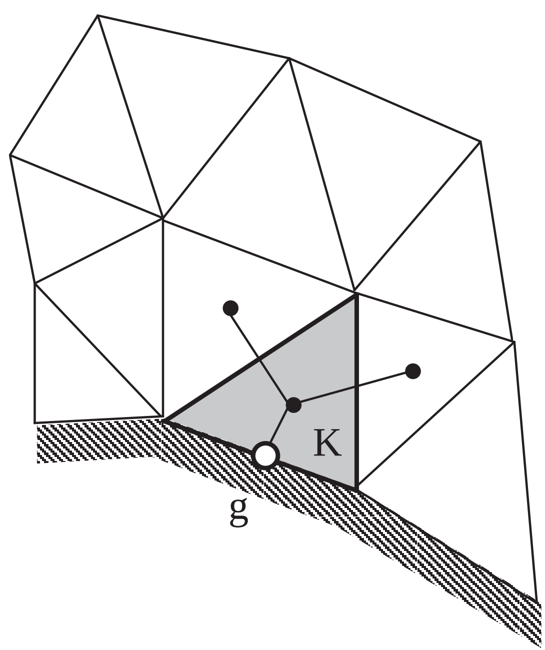

Figure 4.8 *Slope limitation at a solid boundary.*

For the face(s) $g \subset \partial K$ lying on the solid boundary, we determine preliminary values $\overline{z_{K,\,g}}$ by taking in consideration at this level the nonpenetrability boundary condition (3.5.1). We introduce the two components n_g^x and n_g^y of the normal $\mathbf{n}_g$ at the boundary and we set, in coherence with variables (4.4.8) :

$$(4.4.21)\qquad \left\{ \begin{array}{rcl} \overline{\rho_{K,\,g}} & = & \rho_K \\ \overline{\rho_{K,\,g}}\ \overline{u_{K,\,g}} & = & \rho_K \left(u_K - (\mathbf{u}_K \bullet \mathbf{n}_g)\, n_g^x \right) \\ \overline{\rho_{K,\,g}}\ \overline{v_{K,\,g}} & = & \rho_K \left(v_K - (\mathbf{u}_K \bullet \mathbf{n}_g)\, n_g^y \right) \\ \overline{p_{K,\,g}} & = & p_K \,. \end{array} \right.$$

We consider also these values for the limitation algorithm. We define "external values" z_L for $L = g$ and face g lying on the boundary as equal to the ones defined in relation (4.4.21) :

(4.4.22) $\quad z_g \equiv \overline{z_{K,\,g}}\,, \quad z(\bullet)$ field defined in (4.4.21), $g \subset \partial K$ on the boundary.

Then the extrapolation algorithm that conducts to relation (4.4.15) for extrapolated values $z_{K,\,f}$ is used as in the internal case.

• When all values $z_{K,\,f}$ are known for all control volumes $K \in \mathcal{E}_{\mathcal{T}}$, all faces $f \subset K$ and all fields $z(\bullet)$ defined at relation (4.4.8), extrapolated states $W_{K,\,f}$ are naturally defined by going back to the conservative variables. Then we introduce these states as arguments of the flux function $\Phi\big(\bullet\,,\, \mathbf{n}_f\,,\, \bullet\big)$ and obtain by this way a new system of ordinary differential equations :

$$(4.4.23)\quad |K|\ \frac{\mathrm{d}W_K}{\mathrm{d}t} + \sum_{f \subset \partial K} |f|\ \Phi\big(W_{K,\,f}\,,\, \mathbf{n}_f\,,\, W_{K_r(f),\,f}\big) \;=\; 0\,, \qquad K \in \mathcal{E}_{\mathcal{T}}\,.$$

The numerical integration of such kind of system is just a question of Runge-Kutta scheme as presented in [CDV92]. We have used with success in [DM92] the Heun scheme of second order accuracy for discrete integration of (4.4.23) between time steps $n\,\Delta t$ and $(n+1)\,\Delta t$:

$$(4.4.24)\ \frac{|K|}{\Delta t}\left(\widetilde{W_K} - W_K^n\right) + \sum_{f \subset \partial K} |f|\ \Phi\Big(W_{K,\,f}^n\,,\, \mathbf{n}_f\,,\, W_{K_r(f),\,f}^n\Big) \;=\; 0\,,\ K \in \mathcal{E}_{\mathcal{T}}$$

$$(4.4.25)\ \frac{|K|}{\Delta t}\left(\widetilde{\widetilde{W_K}} - \widetilde{W_K}\right) + \sum_{f \subset \partial K} |f|\ \Phi\Big(\widetilde{W}_{K,\,f}\,,\, \mathbf{n}_f\,,\, \widetilde{W}_{K_r(f),\,f}\Big) \;=\; 0\,,\ K \in \mathcal{E}_{\mathcal{T}}$$

$$(4.4.26)\quad W_K^{n+1} \;=\; \frac{1}{2}\left(\widetilde{\widetilde{W_K}} \,+\, W_K^n\right), \quad K \in \mathcal{E}_{\mathcal{T}}\,.$$

• *Acknowledgments.* The author thanks Edwige Godlewski and Stéphanie Mengué for helpfull comments on the preliminary edition of this work.

5) References.

[Au84] J. Audounet. Solutions discontinues paramétriques des systèmes de lois de conservation et des problèmes aux limites associés, *Séminaire*, Université Toulouse 3, 1983-84.

[BLN79] C. Bardos, A.Y. Leroux, J.C. Nédélec. First Order Quasilinear Equations with Boundary Conditions, *Comm. Part. Diff. Eqns.*, vol. 4, p. 1017-1034, 1979.

[Be86] A. Benabdallah. Le "p-système" sur un intervalle, *Comptes Rendus de l'Académie des Sciences,* Paris, tome 303, Série 1, p. 123-126, 1986.

[BS87] A. Benabdallah, D. Serre. Problèmes aux limites pour les systèmes hyperboliques non-linéaires de deux équations à une dimension d'espace, *Comptes Rendus de l'Acad. des Sciences,* Paris, tome 303, Série 1, p. 677-680, 1987.

[BDM89] F. Bourdel, P. Delorme, P. Mazet. Convexity in Hyperbolic Problems. Application to a Discontinuous Galerkin Method for the Resolution of the Polydimensional Euler Equations, Second International Conference on Hyperbolic Problems (Ballmann-Jeltsch Editors), *Notes on Numerical Fluid Dynamics,* vol. 24, p. 31-42, Vieweg, 1989.

[Ca85] H.B. Callen. *Thermodynamics and an introduction to thermostatics, second edition*, John Wiley & Sons, New York, 1985.

[Ch83] H.B. Callen. S. Chakravarthy. Euler solutions, Implicit schemes and Boundary Conditions, *AIAA Journal*, vol. 21, n° 5, p. 699-706, 1983.

[CDV92] D. Chargy, F. Dubois, J.P. Vila. Méthodes numériques pour le calcul d'écoulements compressibles, applications industrielles, *cours de l'Institut pour la Promotion des Sciences de l'Ingénieur*, Paris, septembre 1992.

[CF48] R. Courant, K.O. Friedrichs. *Supersonic Flow and Shock Waves,* Interscience, New-York, 1948.

[Du87] F. Dubois. Boundary Conditions and the Osher Scheme for the Euler Equations of Gas Dynamics, *Internal Report n⁰ 170 of Centre de Mathématiques Appliquées de l'Ecole Polytechnique,* Palaiseau, September 1987.

[Du88] F. Dubois. Conditions aux limites fortement non linéaires pour les équations d'Euler, in *Cours CEA-EDF-INRIA* "Méthodes de différences finies et équations hyperboliques", (P.L. Lions Editor), november 1988.

[Du90] F. Dubois. Concavity of thermostatic entropy and convexity of Lax's mathematical entropy, *Aerospace Research*, n^0 1990-3, p. 77-80, may 1990.

[Du91] F. Dubois. Nonlinear Interpolation and Total Variation Diminishing Schemes, *Third International Conference on Hyperbolic Problems*, (Engquist-Gustafsson Editors), Chartwell-Bratt, p. 351-359, 1991.

[Du98] F. Dubois. Tout ce que vous avez toujours voulu savoir sur les conditions aux limites sans jamais oser le demander, *Seminar, Centre de Mathématiques Appliquées, Ecole Polytechnique*, Palaiseau, juin 1998.

[Du99] F. Dubois. Flux limite de paroi mobile, *Internal Report* n^0 328/99, CNAM-IAT, october 1999.

[DLL91] F. Dubois, B. Larrouturou, A. Lerat. Résolution des équations d'Euler multidimensionnelles, *Cours au DEA de l'Université Paris 6*, juin 1991.

[DLF87] F. Dubois, P. Le Floch. Condition à la limite pour un système de lois de conservation, *Comptes Rendus de l'Académie des Sciences,* Paris, tome 304, Série 1, p. 75-78, 1987.

[DLF88] F. Dubois, P. Le Floch. Boundary Conditions for Nonlinear Hyperbolic Systems of Conservation Laws, *Journal of Differential Equations,* vol. 71, n^{o}1, p. 93-122, 1988.

[DLF89] F. Dubois, P. Le Floch. Boundary Conditions for Nonlinear Hyperbolic Systems of Conservation laws, Second International Conference on Hyperbolic Problems (Ballmann-Jeltsch Editors), *Notes on Numerical Fluid Dynamics,* vol. 24, p. 96-104, Vieweg, 1989.

[DM92] F. Dubois, O. Michaux. Solution of the Euler Equations Around a Double Ellipsoïdal Shape Using Unstructured Meshes and Including Real Gas Effects, *Workshop on Hypersonic Flows for Reentry Problems*, (Désidéri-Glowinski-Périaux Editors), Springer Verlag, vol. II, p. 358-373, 1992.

[FGH91] I. Faille, T. Gallouët, R. Herbin. Les Mathématiciens découvrent les Volumes Finis, *Matapli, n^o 23, p. 37-48,* octobre 1991.

[FL71] K.O. Friedrichs, P.D. Lax. Systems of conservation equations with a convex extension, *Proc. Nat. Acad. Sci. U.S.A.*, vol. 68, n^{o}8, p. 1686-1688, 1971.

[GB53] P. Germain, R. Bader. Unicité des écoulements avec chocs dans la mécanique de Burgers, *Note technique ONERA OA* n^0 1/1711-1, mai 1953.

[Gi94] M. Gisclon. *Conditions aux limites et approximation parabolique*, PhD thesis, Ecole Normale Supérieure de Lyon, 1994.

[GR96] E. Godlewski, P.-A. Raviart. *Numerical Approximation of Hyperbolic Systems of Conservation Laws*, Applied Mathematical Sciences, vol. 118, Springer, New York, 1996.

[Go59] S.K. Godunov. A Difference Method for the Numerical Computation of Discontinuous Solutions of the Equations of Fluid Dynamics, *Math. Sbornik*, vol. 47, p. 271-290, 1959.

[Go61] S.K. Godunov. An intersting class of quasilinear Systems, *Doc. Akad. Nauk. SSSR*, vol. 139, p. 521-523 and *Soviet Math.*, vol. 2, p. 947-949, 1961.

[GZIKP79] S.K. Godunov, A. Zabrodine, M. Ivanov, A. Kraiko, G. Prokopov. *Résolution numérique des problèmes multidimensionnels de la dynamique des gaz,* Editions de Moscou, 1979.

[Ha83] A. Harten. High Resolution Schemes for Hyperbolic Conservation Laws, *Journal of Computational Physics,* vol. 49, p. 357-393, 1983.

[HLV83] A. Harten, P.D. Lax, B. Van Leer. On Upstream Differencing and Godunov-type Schemes for Hyperbolic Conservation Laws, *SIAM Review,* vol. 25, n^0 1, p. 35-61, January 1983.

[Hi86] R.L. Higdon. Initial-Boundary Value Problems for Linear Hyperbolic Systems, *SIAM Review*, vol. 28, n^0 2, p. 177-217, 1986.

[Kr70] H.O. Kreiss. Initial Boundary Value Problems for Hyperbolic Systems, *Comm. Pure Applied Math.*, vol. 23, p. 277-298, 1970.

[Kv70] S. Kruzkov. First Order Quasi-Linear Systems in Several Independent Variables, *Math. Sbornik*, vol. 123, p. 228-255, et *Math. USSR Sb.*, vol. 10, n^0 2, p. 217-243, 1970.

[LL54] L. Landau, E. Lifchitz. *Fluid Mechanics*, 1954, Editions de Moscou, 1967.

[Lax73] P.D. Lax. Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves, *Conf. Board in Mathematical Sciences,* vol. 11, SIAM, Philadelphia, 1973.

[Li77] T.P. Liu. Initial-Boundary Value Problems for Gas Dynamics, *Archive for Rational Mechanics and Analysis*, vol. 64, p. 137-168, 1977.

[Li82] T.P. Liu. Transonic Gas Flow in a Duct of Varying Area, *Archive for Rational Mechanics and Analysis*, vol. 80, n^0 1, p. 1-18, 1982.

[MO75] A. Majda, S. Osher. Initial-Boundary Value Problems for Hyperbolic Equations with Uniformly Characteristic Boundaries, *Comm. Pure Applied Math.*, vol. 28, p. 607-675, 1975.

[MBGB87] P. Mazet, F. Bourdel, R. Greborio, J. Borée. Application de la méthode variationnelle d'entropie à la résolution des équations d'Euler, *ONERA, Centre d'Etudes et de Recherches de Toulouse, Internal Report*, 1987.

[NS77] T. Nishida, J. Smoller. Mixed Problems for Nonlinear Conservation Laws, *Journal of Differential Equations,* vol. 23, p. 244-269, 1977.

[Ol57] O. Oleinik. Discontinuous Solutions of Nonlinear Differential Equations, *Usp. Mat. Nauk. (N.S.)*, vol. 12, p. 3-73, and *Amer. Math. Transl.*, Ser. 2, vol. 26, p. 95-172, 1957.

[OC83] S. Osher, S. Chakravarthy. Upwind Schemes and Boundary Conditions with Applications to Euler Equations in General Geometries, *Journal of Computational Physics*, vol. 50, p. 447-481, 1983.

[Pa80] S.V. Patankar. *Numerical Heat Transfer and Fluid Flow*, Hemisphere publishing, 1980.

[Po88] M. Pollet. Méthodes de calcul relatives aux interfaces missiles-propulseurs, *Internal report*, Aerospatiale Les Mureaux, 1988.

[Ro72] P.J. Roache. *Computational Fluid Dynamics*, Hermosa Publishers, Albukerque, 1972.

[Se96] D. Serre. *Systèmes de lois de conservation.* vol. 1. *Hyperbolicité, entropies, ondes de choc.* vol. 2. *structures géométriques, oscillations et problèmes mixtes*, Diderot, Paris, 1996.

[Sm83] J. Smoller. *Shock Waves and Reaction-Diffusion Equations*, Springer Verlag, Berlin, 1983.

[VL77] B. Van Leer. Towards the Ultimate Conservative Difference Scheme IV. A New Approach to Numerical Convection, *Journal of Computational Physics,* vol. 23, p. 276-299, 1977.

[VL79] B. Van Leer. Towards the Ultimate Conservative Difference Scheme V. A Second Order Sequel to Godunov's Method, *Journal of Computational Physics,* vol. 32, n° 1, p. 101-136, 1979.

[VV78] H. Viviand, J.P. Veuillot. Méthodes pseudo-instationnaires pour le calcul d'écoulements transsoniques, *Publication ONERA* n° 1978-4, 1978.

[YBW82] H. Yee, R. Beam, R. Warming. Boundary Approximations for Implicit Boundary Approximations for Implicit Schemes for One-Dimensional Inviscid Equations of Gas Dynamics, *AIAA Journal*, vol. 20, p. 1203-1211, 1982.